Combinatorics: Exam Review

Cheat Sheet

	Repeats allowed?	Order matters?	Formula
Permutation n items to choose from Select r items	no	yes	$P(n,r) = \frac{n!}{(n-r)!}$
Set n items to choose from Select r items	no	no	$C(n,r) = \frac{n!}{r!(n-r)!}$
Ordered list n items to choose from Select r items	yes	yes	n^r
Unordered list n different types Select r items	yes	no	$C(n+r-1,r)$
Unordered list for Binary Strings r 1's $n-r$ 0's	yes	no	$C(n,r)$

The Rule of Sums: If there are A ways of doing one thing, and B ways of doing another, then the total amount of ways you can do "A or B" is $A+B$.

The Rule of Sums with Overlap: If there are A ways of doing one thing, and B ways of doing another, but there is an overlap of C, then the total amount of ways you can do "A or B" is $A+B-C$.

The Rule of Products: If there are A ways of doing one thing, and B ways of doing another, then the total amount of ways you can do "A and B" is $A \cdot B$.

The Rule of Complements: If there are C total ways to perform some action, and A total ways meet some criteria, then the total ways that *don't* meet that criteria is $A' = C - A$. In other words, $A + A' = C$

By Rachel Singh, last updated December 6, 2018

Ordered List Problems

Order matters, repetitions allowed

Question 1

In an arcade cabinet, the high score table lets you enter 3 characters. You can choose either letters (A-Z) or numbers (0-9), so there are 36 options. How many ways are there to make a high score name?

Question 2

For some state, the license plate requires three letters (A-Z is 26 letters), followed by three numbers (0-9 is 10 numbers). How many different ways are there to build a license plate if you must have 3 letters followed by 4 numbers?

Question 3

In a poorly-designed website, your password must be 5 characters, and can be either all capital letters (A-Z), or all lower-case letters (a-z), OR all numbers, but not a mix. How many ways are there to build a password?

Question 4

If there are 4 types of muffins available, and you're buying one muffin for you and one muffin for a friend, how many ways can you select muffins?

Question 5

In C++, variable names can contain lower case letters (`a-z`, 26 options), upper case letters (`A-Z`, 26 options), and numbers (`0-9`, 10 options), and underscores (_, 1 option). Altogether, this is 26+26+10+1 = 63 options.
How many ways are there to create a variable name of length 5 if...

a. There's no restriction.

b. The variable name *cannot* begin with a number, but it can begin with any other character and numbers can occur in all following characters.

c. The variable name has letters, numbers, and underscores, but it can only use all UPPERCASE or all lowercase.

Solutions

1. 3 characters to enter. 26 letters, 10 numbers, 36 total options. This is an ordered list: Order matters ("CAT" and "ACT" are different) and duplicates are allowed ("AAA"). Result: 36^3

2. Letters: 26^3, Numbers: 10^4, all ways to make license plates: $26^3 \cdot 10^4$

3. All capitals: 26^5, All lower: 26^5, All numbers: 10^5. All ways to build a password: $26^5 + 26^5 + 10^5$

4. 4 types of muffins. One for you, one for friends. This is an ordered list; (YOU, FRIEND). Result: 4^2

5. 26 lower, 26 upper, 10 numbers, 1 underscore (63 total options).
a. No restrictions on variable name (except length 5): 63^5

b. Cannot begin with a number: letters/underscore AND then unrestricted... $53^1 \cdot 63^4$

c. All upper: 26+10+1 options, All lower: 26+10+1 options. Result: $37^5 + 37^5$

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Permutation Problems

Order matters, repetitions not allowed

Question 6
How many ways are there to line up 5 kids in a queue?

Question 7
How many ways are there to rearrange the letters in the word "CATS"?

Question 8
In a class of 20 students, a class president, vice president, and secretary must be elected. Assuming nobody can serve more than one role, how many ways are there to select the president, VP, and secretary?

Question 9
In a classroom, there are 10 computer science students, 8 IT students, and 5 math students. A president, vice president, and secretary must be elected. How many ways are there to elect these positions, with the additional constraints:

a. All three officers must be of the same major

b. At least one officer must be a math student

c. Either the president or the vice president must be an IT student (but not both), and CS and Math will be chosen for the other roles.

Question 10
A combination lock has three columns, and each column has numbers 0 through 9 on it.

a. How many different combinations are there?

b. How many combinations will have no duplicate numbers?

c. How many combinations have the first and second number matching?

d. How many combinations have exactly two of three numbers matching?

Solutions

6. 5 kids to choose, select 5: $P(5, 5)$

7. four letters, place all 4: $P(4, 4)$

8. 20 students, 3 positions: $P(20, 3)$

9. 10 CS, 8 IT, 5 Math (23 total).
a. All officers are same major. All CS: $P(10, 3)$, All IT: $P(8, 3)$, All Math: $P(5, 3)$.
Result: $P(10, 3) + P(8, 3) + P(5, 3)$.

b. At least one officer must be a math student: Use rule of complements.
No restrictions: $P(23, 3)$. No math: $P(18, 3)$. Result: $P(23, 3) - P(18, 3)$.

c. Either pres or VP is an IT student (but not both).
Pres is IT and VP/Sec is other: $P(8, 1) \cdot P(15, 2)$
VP is IT and Pres/Sec is other: $P(8, 1) \cdot P(15, 2)$
Result: $P(8, 1) \cdot P(15, 2) + P(8, 1) \cdot P(15, 2)$

Set Problems

Order doesn't matter, repetitions not allowed

Question 11

In a class of 20 students, how many ways can a 3-person committee be formed?

Question 12

In a classroom, there are 10 computer science students, 8 IT students, and 5 math students. A committee of 4 people must be chosen. How many ways are there to select members, with the additional constraints:

a. How many ways can there be the same amount of CS students and IT students?

b. How many ways can there be more CS students than other students?

Question 13

A box contains 10 toys. 2 toys in the box are broken. Let's say a kid will take 3 toys from the box...

a. How many ways are there for the kid to select 3 toys?

b. How many ways will there be exactly 1 broken toy?

c. How many ways will contain all the broken toys?

d. How many ways will contain no broken toys?

Question 14

In a jar, there are 10 red, 5 green, and 5 blue marbles.

a. How many ways can 4 marbles be selected, without replacement and without regard to order?

b. Of these selections, how many have all 4 be the same color?

c. How many have 2 of one color and 2 of a second color?

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Solutions

11. 20 students, 3 person committee. $C(20,3)$

12. 10 CS, 8 IT, 5 Math (23 total). 4 person committee.
a. Same amount of CS and IT...
   2 CS, 2 IT: $C(10,2) \cdot C(8,2)$
   1 CS, 1 IT, 2 Math: $C(10,1) \cdot C(8,1) \cdot C(5,2)$
   0 CS, 0 IT, 4 Math: $C(5,4)$
   Result: $C(10,2) \cdot C(8,2) + C(10,1) \cdot C(8,1) \cdot C(5,2) + C(5,4)$

b. More CS than others...
   4 CS, 0 others: $C(10,4)$
   3 CS, 1 others: $C(10,3) \cdot C(13,1)$
   Result: $C(10,4) + C(10,3) \cdot C(13,1)$

13. 10 toys, 2 broken, 8 not broken, take 3 toys from box.
a. No restrictions: $C(10,3)$

b. Exactly 1 broken toy: $C(2,1) \cdot C(8,2)$

c. All broken toys: $C(2,2) \cdot C(8,1)$

d. No broken toys: $C(8,3)$

14. 10 red marbles, 5 green marbles, 5 blue marbles (20 total).
a. No restrictions: $C(20,4)$

b. All 4 same color:
   All red: $C(10,4)$
   All green: $C(5,4)$
   All blue: $C(5,4)$
   Result: $C(10,4) + C(5,4) + C(5,4)$

c. 2 of one color and two of another color:
   2 red 2 blue: $C(10,2) \cdot C(5,2)$
   2 red 2 green: $C(10,2) \cdot C(5,2)$
   2 green 2 blue: $C(5,2) \cdot C(5,2)$
   Result: $C(10,2) \cdot C(5,2) + C(10,2) \cdot C(5,2) + C(5,2) \cdot C(5,2)$

Unordered List Problems

Order doesn't matter, repetitions allowed

Binary sequence theorem
The number of binary sequences with r 1's and $n-r$ 0's is $C(n,r)$.

Question 15
How many binary sequences can be generated of length 3 that have exactly 2 ones? Solve by writing out all possibilities AND using the Binary Sequence Theorem.

Question 16
How many binary sequences can be generated of length 8 that have exactly 6 ones?

Question 17
You're purchasing a bag of 12 pieces of fruit from a store that sells 6 types of fruit. How many ways can you buy 12 pieces of fruit?
Remember: 1's are separators and 0's are fruits. How many 0's and 1's?s

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Question 18
How many ways are there to buy 20 fruits at a store that sells 3 types of fruits?

Question 19
How many ways are there to buy 20 fruits at a store that sells 4 types of fruits?

Question 20
How many ways are there to buy 20 fruits at a store that sells 30 types of fruits?

Solutions

15. Binary sequence, length 3, exactly 2 ones (so 1 zero): $r = 2$, $n - r = 1$, $n = 3$, Result: $C(3, 2)$.

16. Binary sequence, length 8, exactly 6 ones (so 2 zeroes): $r = 6$, $n - r = 2$, $n = 8$, Result: $C(8, 6)$.

17. 12 pieces of fruit to buy, 6 types of fruit to choose from.

Binary sequence method:
0's are fruits, 1's are separators. There are 6 types of fruit, so 5 separators.
$r = 5$, $n - r = 12$, $n = 17$. Result: $C(17, 5) = 6188$

Other method with $C(n + r - 1, r)$ way:
Buying 12 pieces $r = 12$, 6 types to choose from, $n = 6$.
$C(12 + 6 - 1, 6) = C(17, 12) = 6188$

18. 20 fruits, 3 types: $r = 2, n - r = 20, n = 22$ $C(22, 2) = 231$

19. 20 fruits, 4 types: $r = 3, n - r = 20, n = 23$ $C(23, 3) = 1771$

20. 20 fruits, 30 types: $r = 29, n - r = 20, n = 49$ $C(49, 29) = 28, 277, 527, 346, 376$

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