Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group.**

Names:

Combinatorics: Recurrence Relations

Review: Closed & Recursive formulas

Remember chapter 1.2...?
Back in the first week of CS 210, you were given sequences of numbers... let's say

$$3, 5, 7, 9, 11$$

...and you tasked with coming up with **closed formulas** and **recursive formulas** for these.

Closed formulas were based only on the value of n, such as...

$$a_n = 2n + 1$$

And recursive formulas were based on some starting value, a_1, with each subsequent element being based off a previous element., such as...

$$a_1 = 3; \qquad a_n = a_{n-1} + 2$$

A full class and 4.4 chapters later, we will actually show you *how* to come up with a formula given a sequence of numbers... No guesswork required.

By Rachel Singh, last updated October 1, 2018

Difference tables

Finding formulas: Recursive

Find a recursive formula for the sequence $2, 5, 8, 11, 14, ...$

First, we will be assigning this sequence of numbers to a variable s, where n is the index (or position) in the sequence, and s_n is the element (or term) at that position.

Previously, we had our formulas begin at a_1, but now we will be starting our sequences at index 0, like when programming.

First let's build a table with n, the index, s_n, the element at that index, and $\triangle_n$, the difference between two elements ($\triangle_n = s_{n+1} - s_n$)

index n	0	1	2	3	4
element at n s_n	2	5	8	11	14
different between elements $\triangle_n$	3	3	3	3	3

With the table, we can see the difference between each term (2 to 5, 5 to 8, etc...) is **3**. With this information, we can write any value s_1 in terms of the previous value:

$$s_1 = s_0 + \triangle_0 \qquad (5 = 2 + 3)$$

$$\begin{aligned} s_2 &= s_1 + \triangle_1 && (8 = 5 + 3) \\ &= s_0 + (\triangle_0 + \triangle_1) && (8 = 2 + 3 + 3) \end{aligned}$$

$$\begin{aligned} s_3 &= s_2 + \triangle_2 && (11 = 8 + 3) \\ &= s_0 + (\triangle_0 + \triangle_1 + \triangle_2) && (11 = 2 + 3 + 3 + 3) \\ &= s_0 + \sum_{k=0}^{2} \triangle_k && (11 = 2 + \sum_{k=0}^{2} 3) \end{aligned}$$

So given the first term being

$$s_0 = 2,$$

we can say the recursive formula is

$$s_n = s_{n-1} + \triangle_{n-1} \qquad \text{or} \qquad s_n = s_{n-1} + 3$$

(Continued) Finding formulas: Closed

But what about the closed formula? Well with the differences we can also describe any term as the *first term* plus the sum of the differences...

$$s_1 = s_0 + \triangle_0 \qquad (5 = 2 + 3)$$

$$\begin{aligned} s_2 &= s_1 + \triangle_1 && (8 = 5 + 3) \\ &= s_0 + (\triangle_0 + \triangle_1) && (8 = 2 + 3 + 3) \end{aligned}$$

$$\begin{aligned} s_3 &= s_2 + \triangle_2 && (11 = 8 + 3) \\ &= s_0 + (\triangle_0 + \triangle_1 + \triangle_2) && (11 = 2 + 3 + 3 + 3) \\ &= s_0 + \textstyle\sum_{k=0}^{2} \triangle_k && (11 = 2 + \textstyle\sum_{k=0}^{2} 3) \end{aligned}$$

So $s_1 = 2 + 3$, $s_2 = 2 + 3 + 3$, $s_3 = 2 + 3 + 3 + 3...$
which we can write as

$$s_n = \sum_{k=0}^{n-1} (3) + 2$$

OR

$$s_n = 3 \cdot n + 2$$

Theorem 1: Fundamental Theorem of Sums and Differences

For any sequence $\{s_n\}$ with first differences $\triangle_k = s_{k+1} - s_k$, and any $n \geq 1$,

$$s_n - s_0 = \sum_{k=0}^{n-1} \triangle_k$$

or

$$s_n = \sum_{k=0}^{n-1} \triangle_k + s_0$$

Question 1
Build a difference table and find the formulas for the sequence: [1]

$$7, 12, 17, 22, 27, ...$$

Question 2
Build a difference table and find the formulas for the sequence: [2]

$$3, 10, 17, 24, 31, ...$$

[1]From Jim Van Horn's POGIL exercises
[2]From Discrete Mathematics, 5.6 exercise 1a, Ensley and Crawley

Complex sequences

Complex sequences

Sometimes, the difference between each term in a sequence is not the same; maybe the difference is 3, then 4, then 5, and so on. In this case, the difference itself also has a difference. In this case, that "difference-of-the-differences" is known as the *second difference*, whereas the difference between the terms themselves is the *first difference*.

Example: Find a closed formula for the sequence 6, 11, 19, 30, 44.

As previously, we can start by writing out a table of the index n, the term s_n, and the difference $\triangle_n$...

n	0	1	2	3	4
s_n	6	11	19	30	44
$\triangle_n$	5	8	11	14	?

Once we've figured out $\triangle_n$, we can see that it isn't constant each time, so we can't apply the same techniques as before. Instead, let's expand the table to have a fourth row: the difference of the differences. We will use a triangle again to symbolize "difference", but we will add a number to it, so $\triangle_n^1$ is the difference between terms, and $\triangle_n^2$ is the difference of those differences.

n	0	1	2	3	4
s_n	6	11	19	30	44
$\triangle_n^1$	5	8	11	14	?
$\triangle_n^2$	3	3	3	...	...

Ahh, can we maybe apply what we learned last time! Solving this will actually mean that we're working recursively.

Let's look at this closer...

n	0	1	2	3	4
s_n	6	11	19	30	44
$\triangle_n^1$	5	8	11	14	?
$\triangle_n^2$	3	3	3	...	...

$$\triangle_n^k = \triangle_{n+1}^{k-1} - \triangle_n^{k-1}$$

(Continued) Complex sequences

$$\begin{array}{lll|l} \triangle_1^1 & = \triangle_0^1 + \triangle_0^2 & & (8 = 5 + 3) \\ \triangle_2^1 & = \triangle_1^1 + \triangle_1^2 & & (11 = 5 + 3 + 3) \\ & = \triangle_0^1 + (\triangle_0^2 + \triangle_1^2) & & (14 = 5 + 3 + 3) \\ \triangle_3^1 & = \triangle_2^1 + \triangle_2^2 & & (11 = 8 + 3) \\ & = \triangle_0^1 + (\triangle_0^2 + \triangle_1^2 + \triangle_2^2) & & (11 = 2 + 3 + 3 + 3) \end{array}$$

So we can use the same strategy to find values of $\triangle_n^1$...

$$\triangle_n^1 = \triangle_0^1 + \sum_{k=0}^{n-1} (\triangle_k^2)$$

$$\triangle_n^1 = 5 + \sum_{k=0}^{n-1} (3)$$

$$\triangle_n^1 = 3n + 5$$

Finding s_n

After we have an equation for the first level difference, we can then repeat the Theorem to find s_n...

$$s_n = s_0 + \sum_{k=0}^{n-1} (\triangle_k^1)$$

$$s_n = 6 + \sum_{k=0}^{n-1} (3k + 5)$$

And simplifying the sum...

$$s_n = 6 + \sum_{k=0}^{n-1} (5) + 3 \sum_{k=0}^{n-1} (k)$$

But how do we find the value of $\sum_{k=0}^{n-1} (k)$?

By Rachel Singh, last updated October 1, 2018

Proposition 1 from Chapter 2.3

$$\sum_{i=1}^{n}(i) = \frac{n(n+1)}{2}$$

Or, rewritten for our use-case:

$$\sum_{k=0}^{n-1}(k) = \frac{n(n-1)}{2}$$

(Continued) Complex sequences

So, continuing to simplify, we have:

$$s_n = 6 + 5n + 3\frac{n(n-1)}{2}$$

$$s_n = 6 + \frac{5n \cdot 2}{2} + \frac{3n(n-1)}{2}$$

$$s_n = \frac{10n + 3n^2 - 3n}{2} + 6$$

$$s_n = \frac{3}{2}n^2 + \frac{7}{2}n + 6$$

And that's the final answer.

Question 3

Build a difference table and find the **closed formula** for the sequence: [3]

$$1, 3, 8, 16, 27, 41, ...$$

[3]From Discrete Mathematics, 5.6 exercise 1c, Ensley and Crawley

0.0.1 Review Theorems

Definition of the kth level difference at index n

$$\triangle_n^k = \triangle_{n+1}^{k-1} - \triangle_n^{k-1}$$

Theorem 1 (Revisited)

$$s_n = \sum_{k=0}^{n-1} \triangle_k^1 + s_0$$

Theorem 2:

$$s_n = s_0 + \sum_{i=1}^{k} \triangle_0^i \cdot C(n,i)$$

Proposition 1 from Chapter 2.3

$$\sum_{k=0}^{n-1}(k) = \frac{n(n-1)}{2}$$