Combinatorics: Rule of Products

Types of structures

	Repeats allowed?	Order matters?	Formula
Permutation n items to choose from Select r items	no	yes	$P(n,r) = \frac{n!}{(n-r)!}$
Set n items to choose from Select r items	no	no	$C(n,r) = \frac{n!}{r!(n-r)!}$
Ordered list n items to choose from Select r items	yes	yes	n^r
Unordered list n different types Select r items	yes	no	$C(n+r-1,n)$
Unordered list for Binary Strings r 1's $n-r$ 0's	yes	no	$C(n,r)$

The Rule of Sums

In combinatorics, the rule of sum or addition principle is a basic counting principle. Stated simply, **it is the idea that if we have A ways of doing something and B ways of doing another thing and we can not do both at the same time, then there are A + B ways to choose one of the actions.** [a]

[a] From https://en.wikipedia.org/wiki/Rule_of_sum

The rule of sums with overlap

If the list to count can be split into two pieces of size x and y, and the pieces have z objects in common, then the original list has $x + y - z$ entries. In terms of sets, we can write this as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ for all sets A and B. [a]

[a] From Discrete Math Mathematical Reasoning and Proofs with Puzzles, Patterns and Games, by Ensley and Crawley

The rule of products

In combinatorics, the rule of product or multiplication principle is a basic counting principle (a.k.a. the fundamental principle of counting). Stated simply, **it is the idea that if there are a ways of doing something and b ways of doing another thing, then there are $a \cdot b$ ways of performing both actions.** [a]

[a] From https://en.wikipedia.org/wiki/Rule_of_product

The rule of complements

If there are x objects, and y of those objects have a particular property, then the number of those objects that do **not** have that particular property is $x - y$. [a]

[a] From Discrete Mathematics, Ensley and Crawley, page 390

Homework

1. In a queue at a movie theater, there are 2 seniors, 5 children, and 3 non-senior adults in line. How many ways can all 10 people line up, if...

   a. There are no restrictions, they can be in any order?
   b. All seniors get to be at the front of the line?
   c. If all seniors OR all children are at the front of the line?

2. A shipment of DVDs has 20 disks in the box. Of these, 3 are defective. We're going to put 5 DVDs up on the display shelf.

   a. How many ways can we select these 5 DVDs, without restriction?
   b. How many ways can we select these 5 DVDs with NO defective DVDs?
   c. How many ways can we select these 5 DVDs, including ALL the defective DVDs?
   d. How many ways can we select these 5 DVDs, with *at least one* defective DVD?

3. On a license plate for some [arbitrary, made up] state, it will have 7 characters on it: First 3 letters (A-Z, 26 options) then 4 numbers (0-9, 10 options). How many ways are there to build a license plate?

4. At a donut shop, they sell 6 types of donuts: Glazed, Chocolate, Cream-filled, Powdered, Cake, and Cinnamon. How many ways can you buy a dozen (12) donuts or two dozen (24) donuts?

5. There are 20 students in a class: 10 CS students, 4 IT students, and 6 math students. We're going to elect a President, Vice President, and Secretary. How many ways are there to elect the three positions, with AT LEAST ONE ROLE filled by a math student?

6. There are 20 students in a class: 10 CS students, 4 IT students, and 6 math students. We're going to elect a committee of 3 people. How many ways are there to elect a committee where AT LEAST ONE member is a math student?

Answer key

1. 10 people: 2 seniors, 5 children, 3 non-senior adults.

   a. $P(10, 10) = 10! = \boxed{3{,}628{,}800}$

   b. $P(2, 2) \cdot P(8, 8) = 2! \cdot 8! = \boxed{80{,}640}$

   c. All seniors in front: $P(2, 2) \cdot P(8, 8)$
   All children in front: $P(5, 5) \cdot P(5, 5)$
   Result: $P(2, 2) \cdot P(8, 8) + P(5, 5) \cdot P(5, 5) = 2! \cdot 8! + 5! \cdot 5! = \boxed{95{,}040}$

2. a. $C(20, 5) = \boxed{15{,}504}$

   b. 20 total DVDs, 3 are bad, so 17 are good... $C(17, 5) = \boxed{6{,}188}$

   c. Select all defectives: $C(3, 3)$ and good DVDs for the rest: $C(17, 2)$
   Result: $C(3, 3) \cdot C(17, 2) = 1 \cdot 136 = \boxed{136}$

   d. Use the rule of complements... Ways to get no bad DVDs + Ways to get at least one bad DVD = Total ways to get DVDs
   $C(20, 5) = C(17, 5) + x$ $x = C(20, 5) - C(17, 5)$ $x = 15,504 - 6,188 = \boxed{9{,}316}$

3. Part 1: 3 letters
   $26^3 = 17,576$

   Part 2: 4 numbers
   $10^4 = 10,000$

   Result: $17,576 + 10,000 = \boxed{27{,}576}$

4. 6 types of donuts, buying 12 or 24 donuts:
   Scenario 1: Buying 12 donuts. 1's are separators, 0's are donuts. Box will have 5 separators and 12 donuts, so we need 17 "spaces" in the box.
   $C(17, 12) = 6,188$

   Scenario 2: Buying 24 donuts. Still 5 separators, but 24 donuts, so 29 total "spaces" in the box:
   $C(29, 24) = 118,755$

   Result: $6,188 + 118,755 = \boxed{124{,}943}$

5. All outcomes:

	Pres	**VP**	**Sec**
3 Math	MATH	MATH	MATH
2 Math	MATH	MATH	IT/CS
2 Math	MATH	IT/CS	MATH
2 Math	IT/CS	MATH	MATH
1 Math	MATH	IT/CS	IT/CS
1 Math	IT/CS	MATH	IT/CS
1 Math	IT/CS	IT/CS	MATH
0 Math	IT/CS	IT/CS	IT/CS

Solving it the long way:

One math role:

math pres, other vp/sec: $P(6,1) \cdot P(14,2) = 1092$

math vp, other pres/sec: $P(6,1) \cdot P(14,2) = 1092$

math sec, other pres/vp: $P(6,1) \cdot P(14,2) = 1092$

Two math roles:

math pres/vp, other sec: $P(6,2) \cdot P(14,1) = 420$

math pres/sec, other vp: $P(6,2) \cdot P(14,1) = 420$

math vp/sec, other pres: $P(6,2) \cdot P(14,1) = 420$

Three math roles:

math pres/vp/sec: $P(6,3) = 120$

Result: $1092 + 1092 + 1092 + 420 + 420 + 420 + 120 = \boxed{4656}$

Solving it with the Rule of Complements:

No restrictions on roles: $P(20,3) = 6840$

NO math roles: $P(14,3) = 2184$

Rule of Complements: Math roles $= 6840 - 2184 = \boxed{4656}$

6. All outcomes:

Math people on committee	**Other people on committee**
0 math	3 IT/CS
1 math	2 IT/CS
2 math	1 IT/CS
3 math	0 IT/CS

Solving it the long way:

1 math and 2 others: $C(6,1) \cdot C(14,2) = 6 \cdot 91 = 546$
2 math and 1 others: $C(6,2) \cdot C(14,1) = 15 \cdot 14 = 210$
3 math: $C(6,3) = 20$

Result: $546 + 210 + 20 = \boxed{776}$

Solving it with the Rule of Complements:

20 students, 10 cs, 4 it, 6 math.
No restriction: $C(20,3) = 1140$
No math: $C(14,3) = 364$

Result: $1140 - 364 = \boxed{776}$