Intro to Combinatorics

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About

In this chapter we are getting introduced to the idea of “counting problems” and what kind of problems we will be looking at.

Topics

1. What is “Combinatorics”?
2. Counting structures
3. Basic practice problems

What is "Combinatorics"?

1. What is “Combinatorics”?

“Combinatorics is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures.”

From https://en.wikipedia.org/wiki/Combinatorics

Notes

Combinatorics = counting problems

1. What is "Combinatorics"?

In this chapter we are concerned with finding out "how many ways can we do [thing]?"

- How many ways are there to elect three positions out of so many candidates?
- How many ways can you get two of the same card-type when drawing two cards?
- etc.

Notes

Combinatorics = counting problems

1. What is "Combinatorics"?

Some problems can be simple, like…

How many ways are there to roll a die?

(In other words, how many different outcomes are there?)

Notes

Combinatorics = counting problems

1. What is "Combinatorics"?

Some problems can be simple, like…

How many ways are there to roll a die?

6 ways

How many ways are there to roll two dice together?

Notes

Combinatorics = counting problems

1. What is "Combinatorics"?

Some problems can be simple, like…

How many ways are there to roll a die?

6 ways

How many ways are there to roll two dice together?

6 ways x 6 ways = 36 ways

Or 36 different outcomes

Notes

Combinatorics = counting problems

Counting Structures

2. Counting Structures

There are different types of structures for different types of counting problems. We differentiate these structures based on whether **repetition is allowed** and whether **order matters**.

Type of structure	Repetition allowed?	Order matters?
Ordered list	**yes**	**yes**
Unordered list	**yes**	**no**
Permutation	**no**	**yes**
Set / Combination	**no**	**no**

Notes

Combinatorics = counting problems

Ordered List
- Repeats? yes
- Order? yes

Unordered List
- Repeats? yes
- Order? no

Permutation
- Repeats? no
- Order? yes

Sets
- Repeats? no
- Order? no

2. Counting Structures – Ordered List

Type of structure	Repetition allowed?	Order matters?
Ordered list	**yes**	**yes**

Ordered list:

- Repetition is allowed (the same outcome can happen multiple times)
- Order matters (different outcomes have different significances… not necessarily their *physical* order.)

Notes

Combinatorics = counting problems

Ordered List
- Repeats? yes
- Order? yes

Unordered List
- Repeats? yes
- Order? no

Permutation
- Repeats? no
- Order? yes

Sets
- Repeats? no
- Order? no

2. Counting Structures – Unordered List

Type of structure	Repetition allowed?	Order matters?
Unordered list	**yes**	**no**

Unordered list:

- Repetition is allowed (the same outcome can happen multiple times)
- Order doesn't matters (doesn't matter if you get A then B, or B then A, those are considered the same)

Notes

Combinatorics = counting problems

Ordered List
- Repeats? yes
- Order? yes

Unordered List
- Repeats? yes
- Order? no

Permutation
- Repeats? no
- Order? yes

Sets
- Repeats? no
- Order? no

2. Counting Structures – Permutation

Type of structure	Repetition allowed?	Order matters?
Permutation	**no**	**yes**

Permutation:

- Repetition is NOT allowed (choosing an item from a pool of options removes it from the pool)
- Order matters (different outcomes have different significances… not necessarily their *physical* order.)

Notes

Combinatorics = counting problems

Ordered List
- Repeats? yes
- Order? yes

Unordered List
- Repeats? yes
- Order? no

Permutation
- Repeats? no
- Order? yes

Sets
- Repeats? no
- Order? no

2. Counting Structures – Sets / Combination

Type of structure	Repetition allowed?	Order matters?
Sets / Combination	**no**	**no**

Permutation:

- Repetition is NOT allowed (choosing an item from a pool of options removes it from the pool)
- Order doesn't matters (doesn't matter if you get A then B, or B then A, those are considered the same)

Notes

Combinatorics = counting problems

Ordered List
- Repeats? yes
- Order? yes

Unordered List
- Repeats? yes
- Order? no

Permutation
- Repeats? no
- Order? yes

Sets
- Repeats? no
- Order? no

2. Counting Structures

Each structure has its own formula – different ways to find the solution to "how many ways…?"

Type of structure	Repetition allowed?	Order matters?	Formula
Permutation	**no**	**yes**	**$P(n,r) = n! / (n-r)!$** For n options, select r
Set / Combination	**no**	**no**	**$C(n,r) = n! / (r! (n-r)!)$** For n options, select r
Ordered list	**yes**	**yes**	**n^r** For n options, select r
Unordered list	**yes**	**no**	**$C(n, r) = n! / (r! (n-r)!)$** For r 1's and n-r 0's OR **$C(n+r-1, r)$** For n options, select r

Notes

Combinatorics = counting problems

Ordered List	**n^r**
- Repeats?	yes
- Order?	yes
Unordered List	**C(n,r)***
- Repeats?	yes
- Order?	no
Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes
Sets	**C(n,r)**
- Repeats?	no
- Order?	no

2. Counting Structures

The hard part of this section is figuring out which formula to use based on the word problem, so we're starting out by looking at *simple problems* for every structure.

Notes

Combinatorics = counting problems

Ordered List	**n^r**
- Repeats?	yes
- Order?	yes
Unordered List	**C(n,r)***
- Repeats?	yes
- Order?	no
Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes
Sets	**C(n,r)**
- Repeats?	no
- Order?	no

Practice Problems

3. Practice Problems

Example 1: You're flipping two coins: A quarter and a penny. How many outcomes are there?

Does order matter?

Order doesn't necessarily mean ***physical order***. In this case, we're making a distinction between the quarter and the penny, so we can consider perhaps the "quarter is first" and the "penny is second". Either way, **order matters here** because the outcomes are different.

Notes

Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes
Sets	**C(n,r)**
- Repeats?	no
- Order?	no
Ordered List	n^r
- Repeats?	yes
- Order?	yes
Unordered List	C(n,r)* Or C(n+r-1, r)
- Repeats?	yes
- Order?	no

3. Practice Problems

Example 1: You're flipping two coins: A quarter and a penny. How many outcomes are there?

Are duplicates allowed?

If we get one heads on a coin flip, does that mean we cannot get heads on a second coin flip? No!

Notes

Permutation — **P(n,r)**
- Repeats? no
- Order? yes

Sets — **C(n,r)**
- Repeats? no
- Order? no

Ordered List — n^r
- Repeats? yes
- Order? yes

Unordered List
C(n,r)* Or C(n+r-1, r)
- Repeats? yes
- Order? no

3. Practice Problems

Example 1: You're flipping two coins: A quarter and a penny. How many outcomes are there?

This is a **Ordered List** because…

- We're making a distinction between a quarter and a penny.
- We can get heads or tails (any outcomes) multiple times.

So we will use the n^r formula.

Notes

Permutation $P(n,r)$
- Repeats? no
- Order? yes

Sets $C(n,r)$
- Repeats? no
- Order? no

Ordered List n^r
- Repeats? yes
- Order? yes

Unordered List
$C(n,r)^*$ Or $C(n+r-1, r)$
- Repeats? yes
- Order? no

3. Practice Problems

Example 1: You're flipping two coins: A quarter and a penny. How many outcomes are there?

This is a **Ordered List** because…

- We're making a distinction between a <u>quarter</u> and a <u>penny</u>.
- We can get <u>heads</u> or <u>tails</u> (any outcomes) multiple times.

So we will use the n^r formula.

n = total possible outcomes for one coin = 2 (heads or tails)
r = amount of times we're flipping coins = 2
result = 2^2 = 4

There are 4 total outcomes for flipping a quarter and a penny:
(H, H), (H, T), (T, H), (T, T)

Notes

<u>Permutation</u> **<u>P(n,r)</u>**
- Repeats? no
- Order? yes

<u>Sets</u> **<u>C(n,r)</u>**
- Repeats? no
- Order? no

<u>Ordered List</u> **<u>n^r</u>**
- Repeats? yes
- Order? yes

<u>Unordered List</u>
<u>C(n,r)*</u> Or **<u>C(n+r-1, r)</u>**
- Repeats? yes
- Order? no

3. Practice Problems

Example 2: You're flipping two coins. How many outcomes are there?

Does order matter?

This question is similar to the last one, except that we aren't differentiating between each coin… so we don't care about order. Getting a "Heads" and then a "Tails" will be considered the same as a "Tails" and then a "Heads".

Notes

Permutation — **P(n,r)**
- Repeats? no
- Order? yes

Sets — **C(n,r)**
- Repeats? no
- Order? no

Ordered List — n^r
- Repeats? yes
- Order? yes

Unordered List
C(n,r)* Or C(n+r-1, r)
- Repeats? yes
- Order? no

3. Practice Problems

Example 2: You're flipping two coins. How many outcomes are there?

Is repetition allowed?

Yes, again, we can have multiple Heads or multiple Tails.

Notes

Permutation — **P(n,r)**
- Repeats? no
- Order? yes

Sets — **C(n,r)**
- Repeats? no
- Order? no

Ordered List — **n^r**
- Repeats? yes
- Order? yes

Unordered List
C(n,r)* Or C(n+r-1, r)
- Repeats? yes
- Order? no

3. Practice Problems

Example 2: You're flipping two coins. How many outcomes are there?

This is an **Unordered List** because…

- Order doesn't matter; we aren't distinguishing between the two coins.
- We can get heads or tails (any outcomes) multiple times.

What are the outcomes in this case?

Notes

Permutation — $P(n,r)$
- Repeats? no
- Order? yes

Sets — $C(n,r)$
- Repeats? no
- Order? no

Ordered List — n^r
- Repeats? yes
- Order? yes

Unordered List — C(n,r)* Or C(n+r-1, r)
- Repeats? yes
- Order? no

3. Practice Problems

Example 2: You're flipping two coins. How many outcomes are there?

This is an **Unordered List** because…

- Order doesn't matter; we aren't distinguishing between the two coins.
- We can get heads or tails (any outcomes) multiple times.

For the ordered list (quarter and penny):

(H, H), (H, T), (T, H), (T, T)

For an unordered list:

{ H,H }, { H, T }, { T, T }

({H,T} and {T,H} are considered the same, so we only list it once)

Notes

Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes
Sets	**C(n,r)**
- Repeats?	no
- Order?	no
Ordered List	**n^r**
- Repeats?	yes
- Order?	yes
Unordered List C(n,r)* Or C(n+r-1, r)	
- Repeats?	yes
- Order?	no

3. Practice Problems

Example 2: You're flipping two coins. How many outcomes are there?

For the formula, we have 2 positions we're choosing (two coins to flip);

$r = 2$

and we have 2 different results we could get (heads/tails);

$n = 2$

So to compute, we use **C(n+r-1, r)** with this information.

$$C(2 + 2 - 1, 2)$$
$$= C(3, 2)$$
$$= 3$$

Notes

Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes
Sets	**C(n,r)**
- Repeats?	no
- Order?	no
Ordered List	n^r
- Repeats?	yes
- Order?	yes
Unordered List C(n,r)* Or C(n+r-1, r)	
- Repeats?	yes
- Order?	no

3. Practice Problems

Example 3: In a class of 10 students, we are electing a president, vice president, and secretary. How many ways are there to elect these positions from the 10 options?

Notes

Permutation	**P(n,r)**
- Repeats?	**no**
- Order?	**yes**
Sets	**C(n,r)**
- Repeats?	**no**
- Order?	**no**
Ordered List	**n^r**
- Repeats?	**yes**
- Order?	**yes**
Unordered List	
C(n,r)* Or C(n+r-1, r)	
- Repeats?	**yes**
- Order?	**no**

3. Practice Problems

Example 3: In a class of 10 students, we are electing a president, vice president, and secretary. How many ways are there to elect these positions from the 10 options?

Does order matter?

Order doesn't necessarily mean ***physical order***. In this case, we're making a distinction between the president, vice president, and secretary. These are different roles.

You can also think of this as it would have to be written in an **ordered way**, like (P, V, S), and not as a set: {A, B, C}

Notes

Permutation — **P(n,r)**
- **Repeats?** no
- **Order?** yes

Sets — **C(n,r)**
- **Repeats?** no
- **Order?** no

Ordered List — n^r
- **Repeats?** yes
- **Order?** yes

Unordered List
C(n,r)* Or C(n+r-1, r)
- **Repeats?** yes
- **Order?** no

3. Practice Problems

Example 3: In a class of 10 students, we are electing a president, vice president, and secretary. How many ways are there to elect these positions from the 10 options?

Is repetition allowed?

It's not explicitly mentioned, but we're going to assume that **if somebody is elected to one position, they cannot fill a second position**. So, no, repetition is not allowed.

Notes

Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes

Sets	**C(n,r)**
- Repeats?	no
- Order?	no

Ordered List	n^r
- Repeats?	yes
- Order?	yes

Unordered List

C(n,r)* Or C(n+r-1, r)

- Repeats?	yes
- Order?	no

3. Practice Problems

Example 3: In a class of 10 students, we are electing a president, vice president, and secretary. How many ways are there to elect these positions from the 10 options?

Order matters, repetition is not allowed.
This is a permutation.

Notes

Permutation P(n,r)
- Repeats? no
- Order? yes

Sets C(n,r)
- Repeats? no
- Order? no

Ordered List n^r
- Repeats? yes
- Order? yes

Unordered List
C(n,r)* Or C(n+r-1, r)
- Repeats? yes
- Order? no

3. Practice Problems

Example 3: In a class of 10 students, we are electing a president, vice president, and secretary. How many ways are there to elect these positions from the 10 options?

We have 10 students to choose from, so n = 10

We have 3 positions to select, so r = 3

The result will be P(10, 3) = 720

$$P(10,3)=\frac{10!}{(10-3)!}=\frac{10!}{(7)!}=\frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{\cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}=10\cdot 9\cdot 8=720$$

Notes

Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes

Sets	**C(n,r)**
- Repeats?	no
- Order?	no

Ordered List	**n^r**
- Repeats?	yes
- Order?	yes

Unordered List

C(n,r)* Or C(n+r-1, r)

- Repeats?	yes
- Order?	no

3. Practice Problems

Example 4: In a class of 10 students, we are electing a committee of 3 people. How many ways are there to elect the committee from the 10 options?

Notes

Permutation	**P(n,r)**
- Repeats?	**no**
- Order?	**yes**
Sets	**C(n,r)**
- Repeats?	**no**
- Order?	**no**
Ordered List	**n^r**
- Repeats?	**yes**
- Order?	**yes**
Unordered List C(n,r)* Or C(n+r-1, r)	
- Repeats?	**yes**
- Order?	**no**

3. Practice Problems

Example 4: In a class of 10 students, we are electing a committee of 3 people. How many ways are there to elect the committee from the 10 options?

Does order matter?

Here, order doesn't matter – we aren't making a distinction between roles on the committee.

If we had ***{ Rai, Anuj, Asha }*** *on a committee,*

it would be considered the same committee as ***{ Asha, Anuj, Rai }***

Notes

Permutation	**P(n,r)**
- Repeats?	**no**
- Order?	**yes**
Sets	**C(n,r)**
- Repeats?	**no**
- Order?	**no**
Ordered List	**n^r**
- Repeats?	**yes**
- Order?	**yes**
Unordered List C(n,r)* Or C(n+r-1, r)	
- Repeats?	**yes**
- Order?	**no**

3. Practice Problems

Example 4: In a class of 10 students, we are electing a committee of 3 people. How many ways are there to elect the committee from the 10 options?

Is repetition allowed?

It's not explicitly mentioned, but we're going to assume that **if somebody is elected to one position, they cannot fill a second position**. So, no, repetition is not allowed.

Notes

Permutation — P(n,r)
- Repeats? no
- Order? yes

Sets — C(n,r)
- Repeats? no
- Order? no

Ordered List — n^r
- Repeats? yes
- Order? yes

Unordered List — C(n,r)* Or C(n+r-1, r)
- Repeats? yes
- Order? no

3. Practice Problems

Example 4: In a class of 10 students, we are electing a committee of 3 people. How many ways are there to elect the committee from the 10 options?

Order doesn't matter, repetition is not allowed.
This is a set.

Notes

Permutation	P(n,r)
- Repeats?	no
- Order?	yes

Sets	C(n,r)
- Repeats?	no
- Order?	no

Ordered List	n^r
- Repeats?	yes
- Order?	yes

Unordered List
C(n,r)* Or C(n+r-1, r)

- Repeats?	yes
- Order?	no

3. Practice Problems

Example 4: In a class of 10 students, we are electing a committee of 3 people. How many ways are there to elect the committee from the 10 options?

We have 10 students to choose from, so n = 10

We have 3 positions to select, so r = 3

The result will be C(10, 3) = 120

$$P(10,3)=\frac{10!}{3!(10-3)!}=\frac{10!}{3!\cdot 7!}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}=120$$

Notes

Permutation	**P(n,r)**
- Repeats?	no
- Order?	yes

Sets	**C(n,r)**
- Repeats?	no
- Order?	no

Ordered List	**n^r**
- Repeats?	yes
- Order?	yes

Unordered List
C(n,r)* Or C(n+r-1, r)

- Repeats?	yes
- Order?	no

Conclusion

You'll get plenty more practice with these concepts going forward.

Next time, we will look at some rules that will help us calculate the outcomes for more sophisticated counting problems.