Solving Recurrence Relations

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About

Back in Chapter 1.2, we worked with sequences of numbers and had to figure out **closed formulas** and **recursive formulas** mostly by trial-and-error.

Now we will look at how to actually, systematically, solve these to find formulas given some sequence of numbers.

Topics

1. Guessing Formulas
2. Difference Tables
3. Complex Sequences

Guessing Formulas

1. Guessing Formulas

How did you try to find formulas for sequences of numbers back in Chapter 1.2?

Maybe you started with some basic sequences that you could identify, like “2, 4, 6, 8” or “3, 6, 9, 12” or even “2, 4, 8, 16, 32” and would check if a sequence followed a similar pattern, but perhaps +/- something else.

$$a_n = 2n + 1$$

$$= 3, 5, 7, 9, \ldots$$

Notes

1. Guessing Formulas

Maybe you would investigate the numbers and see if there was a linear difference between each one…

2 5 8 11 14

+3 +3 +3 +3 …

= It's 3 times n, but offset by -1…

$$a_n = 3n - 1$$

Notes

1. Guessing Formulas

It can be pretty frustrating to try to just analyze a sequence of numbers and hope that you can find the pattern in order to derive a formula.

Instead, how could we solve this in a more reliable way?

Notes

Difference Tables

2. Difference Tables

Let's find a formula for the sequence of numbers: 2, 5, 8, 11, 14

Notes

2. Difference Tables

Let's find a formula for the sequence of numbers: 2, 5, 8, 11, 14

The first step to analyzing a sequence of numbers is to inspect what the differences are between each term. We can build a difference table like this:

Index #	n	0	1	2	3	4
Element at position n	s_n	**2**	**5**	**8**	**11**	**14**
Difference	Δ_n	**3**	**3**	**3**	**3**	**3**

Note that while working with difference tables, we will have the index start at 0 instead of 1.

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

2. Difference Tables

Notice that the difference between each term is 3 in this case.

By recognizing this information, we can essentially *derive* the term value at any arbitrary position *n*.

n Index	0	1	2	3	4
s_n Element	2	5	8	11	14
Δ_n Difference	3	3	3	3	3

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

2. Difference Tables

- $s_0 = 2$.
- $s_1 = 2 + 3$
- $s_2 = 2 + 3 + 3$
- $s_3 = 2 + 3 + 3 + 3$

And so on…

n Index	0	1	2	3	4
s_n Element	2	5	8	11	14
Δ_n Difference	3	3	3	3	3

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

2. Difference Tables

- $s_0 = 2$.
- $s_1 = s_0 + \Delta_0$
- $s_2 = s_0 + \Delta_0 + \Delta_1$
- $s_3 = s_0 + \Delta_0 + \Delta_1 + \Delta_2$

n Index	0	1	2	3	4
s_n Element	2	5	8	11	14
Δ_n Difference	3	3	3	3	3

Let's change these to the corresponding symbols that represent the values.

Now we can see a pattern…

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

2. Difference Tables

- $s_0 = 2$.
- $s_1 = s_0 + \Delta_0$
- $s_2 = s_0 + \Delta_0 + \Delta_1$
- $s_3 = s_0 + \Delta_0 + \Delta_1 + \Delta_2$
- $s_n = s_0 + \sum_{k=0}^{n} \Delta_k$ < Generalized
- $s_n = 2 + 3n$ < We've found a formula

n *Index*	0	1	2	3	4
s_n *Element*	**2**	**5**	**8**	**11**	**14**
Δ_n *Difference*	**3**	**3**	**3**	**3**	**3**

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

2. Difference Tables

This can be further generalized into a theorem for *any* sequence of numbers where every two terms have the same difference between each other…

Theorem 1: Fundamental Theorem of Sums and Differences

From Discrete Mathematics, Ensley & Crawley

For any sequence $\{s_n\}$ with first differences $\Delta_k = s_{k+1} - s_k$, and any $n \geq 1$,

$$s_n - s_0 = \sum_{k=0}^{n-1} \Delta_k$$

Or in other words…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

For sequences with the same difference between each term…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

2. Difference Tables

The theorem works fine for sequences where every two terms have the same difference between them.

In other cases, we have to do a bit more work, but we can eventually use this theorem as well.

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

For sequences with the same difference between each term…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

Complex Sequences

3. Complex Sequences

The theorem works fine for sequences where every two terms have the same difference between them.

In other cases, we have to do a bit more work, but we can eventually use this theorem as well.

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

For sequences with the same difference between each term…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

3. Complex Sequences

Now let's say that we want to find an equation for the sequence:
6, 11, 19, 30, 44.

Again, we build out our difference table.

n *Index*	0	1	2	3	4
s_n *Element*	6	11	19	30	44
Δ_n *Difference*	5	8	11	14	?

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

For sequences with the same difference between each term…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

3. Complex Sequences

We don't have the same difference between each term, so how about the differences between each difference?

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ_n Difference	5	8	11	14	?
Δ_n Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

For sequences with the same difference between each term…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

3. Complex Sequences

Now we'll need to come up with a better labeling scheme for our deltas…

The first-level difference will be represented as

Δ^1

And the second-level difference will be

Δ^2

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Notes

n: Index

s_n: Element at n

Δn: $s_{n+1} - s_n$

For sequences with the same difference between each term…

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences:

Δ^1

2nd level differences:

Δ^2

3. Complex Sequences

We have:

$$\Delta_n^k$$

k = the difference level
n = the index

So... :

- 1^{st} level: $\Delta_n^1 = s_{n+1} - s_n$
- 2^{nd} level: $\Delta_n^2 = \Delta_{n+1}^1 - \Delta_n^1$
- k^{th} level: $\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between $k-1$ levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

Last time, the first-level difference was constant: it was 3 each time.

For the sequence “6, 11, 19, 30, 44”, the first-level difference changes but the second-level difference is constant.

n *Index*	0	1	2	3	4
s_n ***Element***	**6**	**11**	**19**	**30**	**44**
Δ^1_n ***Difference***	**5**	**8**	**11**	**14**	**?**
Δ^2_n ***Difference 2***	**3**	**3**	**3**	**?**	**?**

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

Let's treat this like last time, but looking at 2nd level vs. 1st level…

- $\Delta_0^1 = 5$
- $\Delta_1^1 = 5 + 3$
- $\Delta_2^1 = 5 + 3 + 3$
- $\Delta_3^1 = 5 + 3 + 3 + 3$

n Index	0	1	2	3	4
s_n Element	**6**	**11**	**19**	**30**	**44**
Δ^1_n Difference	**5**	**8**	**11**	**14**	**?**
Δ^2_n Difference 2	**3**	**3**	**3**	**?**	**?**

So we look at the ways to get 5, 8, 11, and 14 (the first level differences)

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

Let's treat this like last time, but looking at 2nd level vs. 1st level…

- $\Delta_0^1 = 5$
- $\Delta_1^1 = \Delta_0^1 + \Delta_0^2$
- $\Delta_2^1 = \Delta_0^1 + \Delta_0^2 + \Delta_1^2$
- $\Delta_3^1 = \Delta_0^1 + \Delta_0^2 + \Delta_1^2 + \Delta_2^2$

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ_n^1 Difference	5	8	11	14	?
Δ_n^2 Difference 2	3	3	3	?	?

And replace the numbers with the general terms.

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

kth level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

Let's treat this like last time, but looking at 2nd level vs. 1st level…

- $\Delta_0^1 = 5$
- $\Delta_1^1 = \Delta_0^1 + \Delta_0^2$
- $\Delta_2^1 = \Delta_0^1 + \Delta_0^2 + \Delta_1^2$
- $\Delta_3^1 = \Delta_0^1 + \Delta_0^2 + \Delta_1^2 + \Delta_2^2$
- $\Delta_n^1 = \Delta_0^1 + \sum_{k=0}^{n-1} \Delta_k^2$

n Index	0	1	2	3	4
s_n Element	**6**	**11**	**19**	**30**	**44**
$\Delta^1{}_n$ Difference	**5**	**8**	**11**	**14**	**?**
$\Delta^2{}_n$ Difference 2	**3**	**3**	**3**	**?**	**?**

Then we can come up with the general form.

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

And with this

$$\Delta_n^1 = \Delta_0^1 + \sum_{k=0}^{n-1} \Delta_k^2$$

we can find our equation for the 1st level difference terms:

$$\Delta_n^1 = 5 + 3n$$

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

But we're still not done because we don't have an equation for the actual terms – s_n – yet.

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

1st level difference:

$$\Delta_n^1 = 5 + 3n$$

And remember Theorem 1:

$$s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$$

Now we can solve for s_n...

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

1^{st} level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$

Start with the Theorem...

n *Index*	0	1	2	3	4
s_n *Element*	6	11	19	30	44
Δ^1_n *Difference*	5	8	11	14	?
Δ^2_n *Difference 2*	3	3	3	?	?

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

1^{st} level difference:

$$\Delta_n^1 = 5 + 3n$$

- $$s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$$
- $$s_n = \sum_{k=0}^{n-1} (5+3k) + s_0$$

Plug in the equation for the first-level difference...

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

1^{st} level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$
- $s_n = \sum_{k=0}^{n-1} (5 + 3k) + s_0$
- $s_n = \sum_{k=0}^{n-1} (5) + 3 \sum_{k=0}^{n-1} (k) + s_0$

Split out the sum to make it easier to solve…

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

1st level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$
- $s_n = \sum_{k=0}^{n-1} (5+3k) + 6$
- $s_n = \sum_{k=0}^{n-1} (5) + 3\sum_{k=0}^{n-1} (k) + 6$
- $s_n = 6 + 5n + 3\sum_{k=0}^{n-1} (k)$

What does the sum come out to now? Time for magic…

n *Index*	0	1	2	3	4
s_n *Element*	**6**	**11**	**19**	**30**	**44**
Δ_n^1 *Difference*	**5**	**8**	**11**	**14**	**?**
Δ_n^2 *Difference 2*	**3**	**3**	**3**	**?**	**?**

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k[th] level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

3. Complex Sequences

1^{st} level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$
- $s_n = \sum_{k=0}^{n-1} (5+3k) + 6$
- $s_n = \sum_{k=0}^{n-1} (5) + 3\sum_{k=0}^{n-1} (k) + 6$
- $s_n = 6 + 5n + 3\sum_{k=0}^{n-1} (k)$

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Random proposition from chapter 2.3 that you totally don't remember.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between k-1 levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Proposition 1 from 2.3:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3. Complex Sequences

1st level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$
- $s_n = \sum_{k=0}^{n-1} (5 + 3k) + 6$
- $s_n = \sum_{k=0}^{n-1} (5) + 3\sum_{k=0}^{n-1} (k) + 6$
- $s_n = 6 + 5n + 3\sum_{k=0}^{n-1} (k)$
- $s_n = 6 + 5n + \frac{3n(n-1)}{2}$

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ^1_n Difference	5	8	11	14	?
Δ^2_n Difference 2	3	3	3	?	?

Change it to...

$$\sum_{k=0}^{n-1} k = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$$

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Proposition 1 from 2.3:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3. Complex Sequences

1st level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = 6 + 5n + \frac{3n(n-1)}{2}$

And we continue simplifying…

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ_n^1 Difference	5	8	11	14	?
Δ_n^2 Difference 2	3	3	3	?	?

Change it to…

$$\sum_{k=0}^{n-1} k = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$$

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k[th] level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Proposition 1 from 2.3:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3. Complex Sequences

1st level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = 6 + 5n + \frac{3n(n-1)}{2}$

- $s_n = \frac{6 \cdot 2}{2} + \frac{5n \cdot 2}{2} + \frac{3n(n-1)}{2}$

Yay, common denominators…

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ_n^1 Difference	5	8	11	14	?
Δ_n^2 Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at n
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Proposition 1 from 2.3:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3. Complex Sequences

1^{st} level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = 6 + 5n + \frac{3n(n-1)}{2}$
- $s_n = \frac{6 \cdot 2}{2} + \frac{5n \cdot 2}{2} + \frac{3n(n-1)}{2}$
- $s_n = \frac{12 + 10n + 3n^2 - 3n}{2}$
- $s_n = \frac{3n^2 + 7n + 12}{2}$

And more simplifying…

n Index	0	1	2	3	4
s_n Element	6	11	19	30	44
Δ_n^1 Difference	5	8	11	14	?
Δ_n^2 Difference 2	3	3	3	?	?

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Proposition 1 from 2.3:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3. Complex Sequences

1st level difference:

$$\Delta_n^1 = 5 + 3n$$

- $s_n = 6 + 5n + \frac{3n(n-1)}{2}$
- $s_n = \frac{6 \cdot 2}{2} + \frac{5n \cdot 2}{2} + \frac{3n(n-1)}{2}$
- $s_n = \frac{12 + 10n + 3n^2 - 3n}{2}$
- $s_n = \frac{3n^2 + 7n + 12}{2}$

n *Index*	0	1	2	3	4
s_n *Element*	6	11	19	30	44
Δ_n^1 *Difference*	5	8	11	14	?
Δ_n^2 *Difference 2*	3	3	3	?	?

$$s_n = \frac{3}{2} n^2 + \frac{7}{2} n + 6$$

And, yes, that's the final answer.

Notes

n: Index
s_n: Element at *n*
Δn: $s_{n+1} - s_n$

Terms s_n with same Δ:

$$s_n = \sum_{k=0}^{n-1} \Delta_k + s_0$$

1st level differences: Δ^1

2nd level differences: Δ^2

k^{th} level difference between *k-1* levels, for n+1 and n:

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Proposition 1 from 2.3:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Conclusion

So, self-evaluation time. How long do you think it would have taken you to come up with

$$s_n = \frac{3}{2}n^2 + \frac{7}{2}n + 6$$

From the sequence 6, 11, 19, 30, 44 ?

Conclusion

Well, at least *now* we know how to figure out a closed formula from a sequence of numbers…