Functions

Written by Rachel Singh, last updated Oct 1, 2018

About

In algebra, we think of functions as something like “f(x)”, where x is the input, it’s plugged into an equation, and we get some output, f(x).

In programming, we can also define functions. These also have inputs and outputs as well.

Let’s look at another way to view functions.

Topics

1. Functions
2. Onto and One-to-one
3. Invertibility Property
4. Additional notation
5. Inverses

1. Functions

1. Functions

Whether we're writing a function in algebra or in a computer program, functions will have inputs and outputs.

We have a **set of all possible inputs** of f, and this set is called the **domain.**

The **set of all possible outputs** of f is called the **codomain.**

Notes

Domain: Set of all possible inputs

Codomain: Set of all possible outputs.

1. Functions

While working with functions in this section, we will use this notation for a function:

$$f : A \to B$$

Where f is the function name, A is the the **domain**, and B is the **codomain**.

The function f associates one input from A with one and only one output in B.

The mapping between A and B is known as the **rule**. It can be a mathematical expression ("$f(x) = x^2$") or even a set of ordered pairs { (a, 1), (b, 2), … }

Notes

Domain: Set of all possible inputs

Codomain: Set of all possible outputs.

$f : A \to B$
function f, with input from set A, & output from set B.

A function f maps some input from A to one and only one output from B.

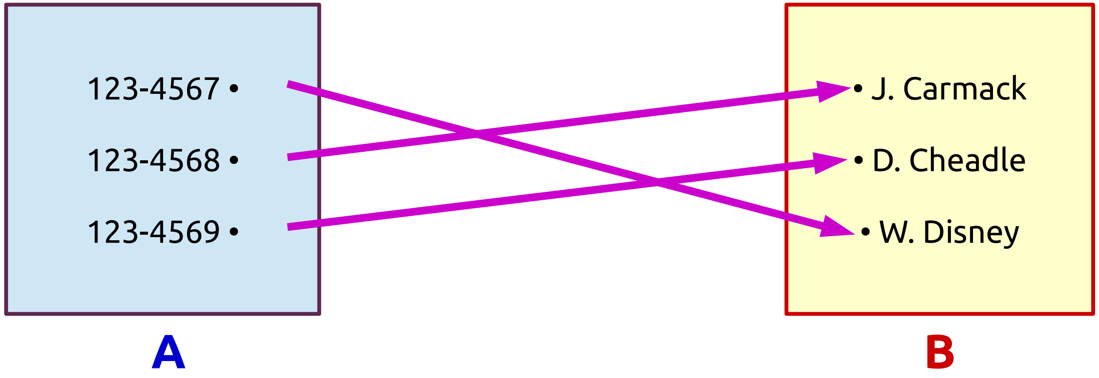
1. Functions

Example: $p : A \rightarrow B$ is a function that maps a phone-number to a single person. Each person has a unique phone number.

A = { 123-4567, 123-4568, 123-4569, etc. }

B = { J. Carmack, D. Cheadle, W. Disney, etc. }

Rule: { (123-4567, J. Carmack), (123-4568, D. Cheadle), (123-4569, W. Disney), … }



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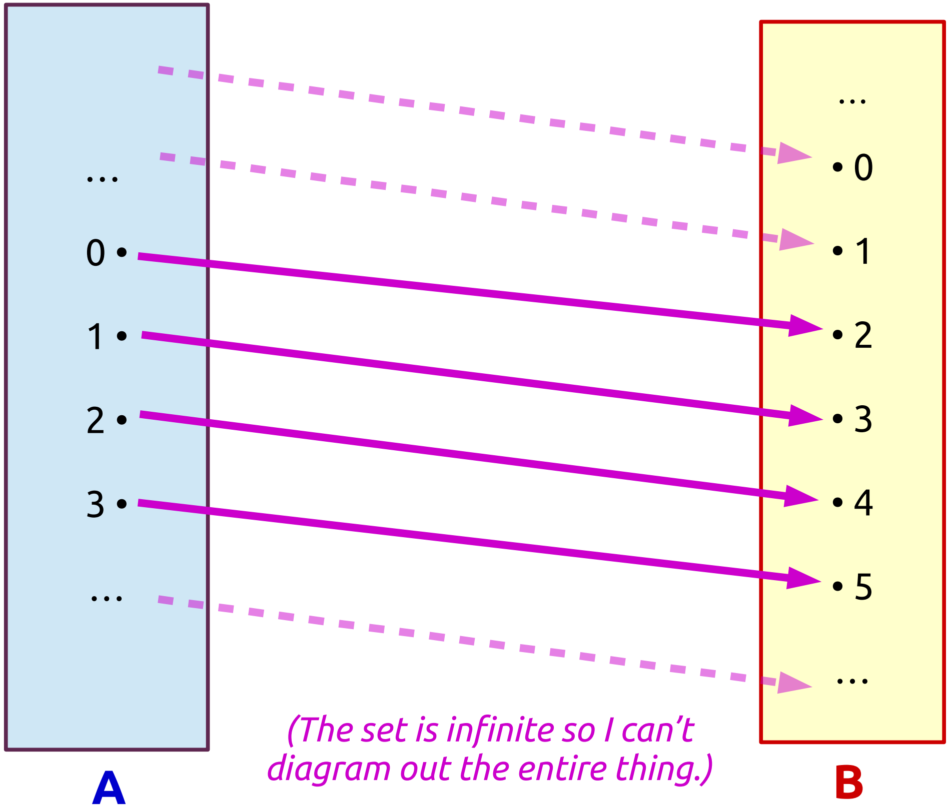
A function f maps some input from A to <u>one and only one</u> output from B.

1. Functions

Example:
$f : \mathbb{Z} \to \mathbb{Z}$ is a function, where the rule is f(x) = x + 2



(The set is infinite so I can't diagram out the entire thing.)

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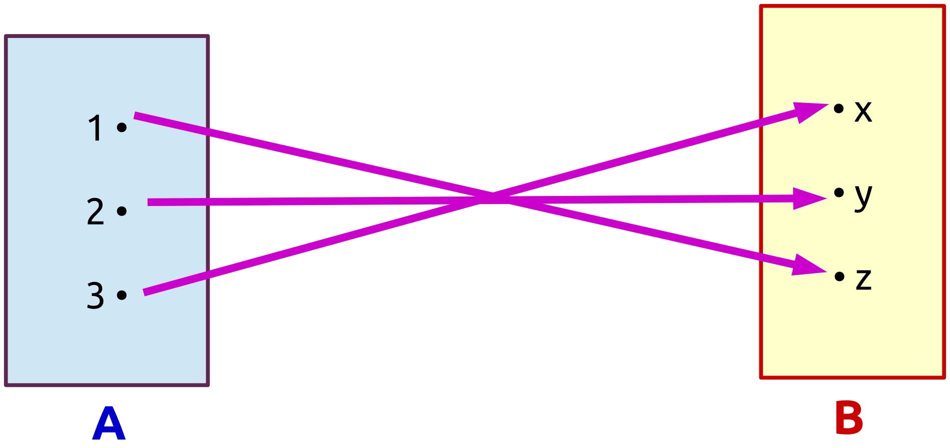
1. Functions

We can also specify the mapping explicitly by using a set of ordered pairs. For example:

Function: $g : A \to B$
A = { 1, 2, 3 }   B = { x, y, z }
Rule: { (1, z), (2, y), (3, x) }



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1. Functions

Practice: Diagram the following function.

Function: $h : X \rightarrow Y$
X = { 2, 4, 6, 8 }
Y = { 1, 3, 5, 7 }
Rule: { (2,7), (4,3), (6,5), (8,1) }

Notes

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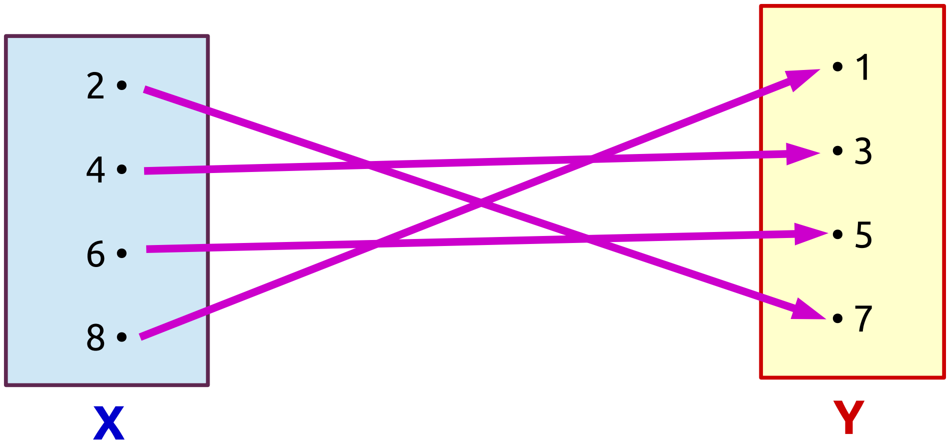

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2. Onto and One-to-one Properties

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We can further be more explicit about what makes a function invertible by identifying the **Onto** and **One-to-one** properties.

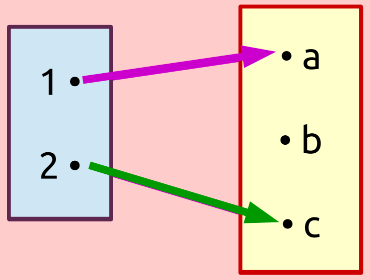
Notes

A function maps each element of the domain to one and only one element of the codomain.

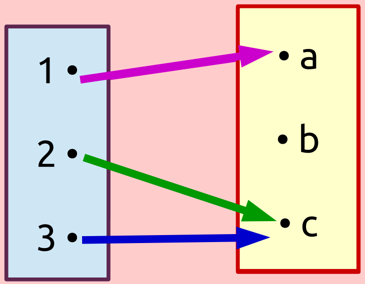
2. Onto and One-to-one Properties

A function is **onto** if every element of the codomain is an output of something from the domain.

In other words, all elements of the codomain are being pointed to by something from the domain.



Not onto



Not onto

Notes

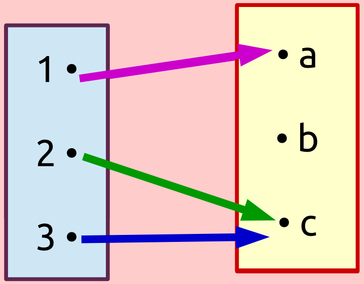
A function maps each element of the domain to one and only one element of the codomain.

Onto: Everything in the codomain is an output of f.

2. Onto and One-to-one Properties

A function is **one-to-one** if nothing in the codomain is the output of two or more inputs from the domain.

This means that no element in the codomain is being pointed to by *more than one* input from the domain.



Not one-to-one

Notes

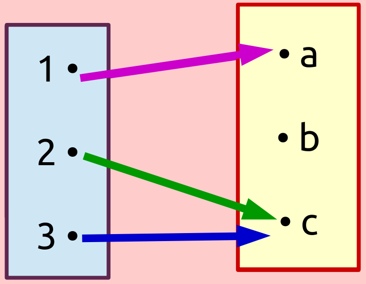
A function maps each element of the domain to one and only one element of the codomain.

Onto: Everything in the codomain is an output of f.

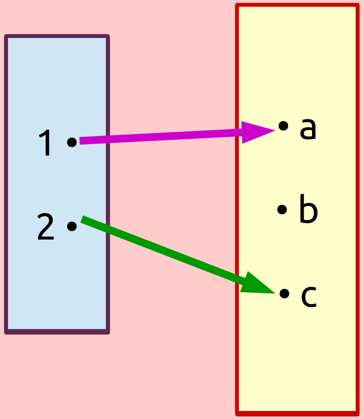
One-to-one: Nothing in the codomain is an output of two or more inputs.

2. Onto and One-to-one Properties

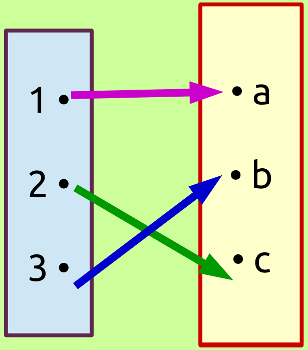
If a function is both **onto and one-to-one**, then it is invertible (the inverse is also a function). If it doesn't have one of these properties, then it is not invertible.



Not one-to-one, inverse is not a function.



Not onto, inverse is not a function.



Onto and one-to-one, inverse is a function.

Notes

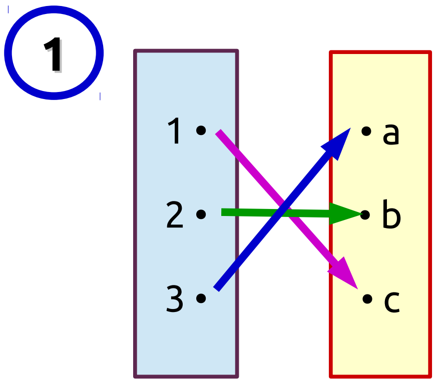
A function maps each element of the domain to one and only one element of the codomain.

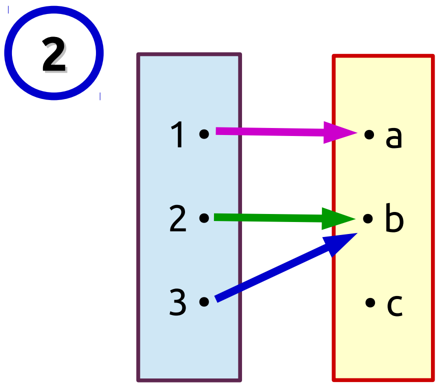
Onto: Everything in the codomain is an output of f.

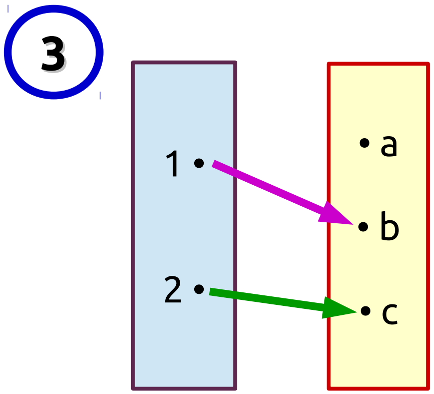
One-to-one: Nothing in the codomain is an output of two or more inputs.

2. Onto and One-to-one Properties

Practice: Identify whether each of these functions are **onto**, **one-to-one**, and **invertible.**







Notes

A function maps each element of the domain to one and only one element of the codomain.

Onto: Everything in the codomain is an output of f.

One-to-one: Nothing in the codomain is an output of two or more inputs.

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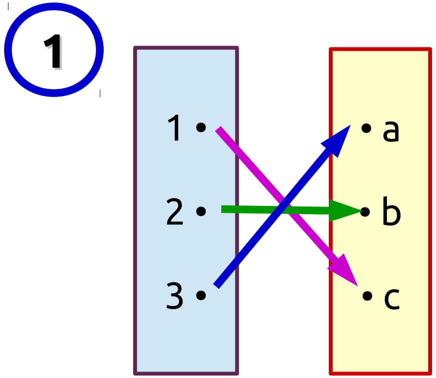
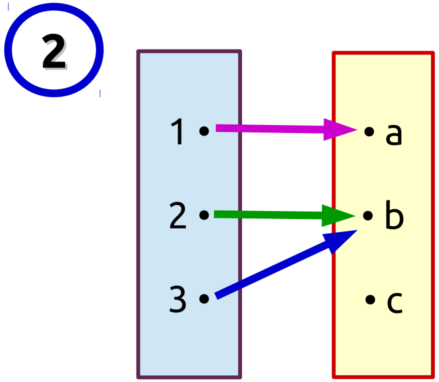
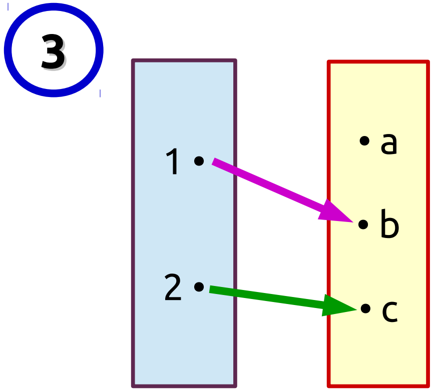
Practice: Identify whether each of these functions are **onto**, **one-to-one**, and **invertible.**



Onto: ✓
One-to-one: ✓
Invertible: ✓



Onto: ✗
One-to-one: ✗
Invertible: ✗



Onto: ✗
One-to-one: ✓
Invertible: ✗

Notes

A function maps each element of the domain to one and only one element of the codomain.

Onto: Everything in the codomain is an output of f.

One-to-one: Nothing in the codomain is an output of two or more inputs.

2. Onto and One-to-one Properties

Practice: Identify whether each of these functions are **onto**, **one-to-one**, and **invertible.**

A = { 1, 2 }, B = { 3, 4 }, C = { 5, 6, 7 }

1. $f : A \to B$ **with the rule, { (1, 3), (2, 4) }**

2. $g : A \to C$ **with the rule, { (1, 5), (2, 6) }**

3. $h : C \to B$ **with the rule, { (5, 3), (6, 3), (7, 4) }**

Notes

A function maps each element of the domain to one and only one element of the codomain.

Onto: Everything in the codomain is an output of f.

One-to-one: Nothing in the codomain is an output of two or more inputs.

2. Onto and One-to-one Properties

Practice: Identify whether each of these functions are **onto**, **one-to-one**, and **invertible.**

A = { 1, 2 }, B = { 3, 4 }, C = { 5, 6, 7 }

1. **$f : A \rightarrow B$ with the rule, { (1, 3), (2, 4) }**

   Onto: ✓
   One-to-one: ✓
   Invertible: ✓

2. **$g : A \rightarrow C$ with the rule, { (1, 5), (2, 6) }**

   The element 7 from the codomain is not being mapped to.

   Onto: ✗
   One-to-one: ✓
   Invertible: ✗

3. **$h : C \rightarrow B$ with the rule, { (5, 3), (6, 3), (7, 4) }**

   The element 3 from the codomain is being mapped to by two inputs, 5 and 6.

   Onto: ✓
   One-to-one: ✗
   Invertible: ✗

Notes

A function maps <u>each element</u> of the domain to <u>one and only one</u> element of the codomain.

Onto: Everything in the codomain is an output of f.

One-to-one: Nothing in the codomain is an output of two or more inputs.

3. Invertibility Property

1. Invertibility Property

Given some function $f : A \rightarrow B$, if the inverse $f^{-1} : B \rightarrow A$ is also a function, then f is invertible.

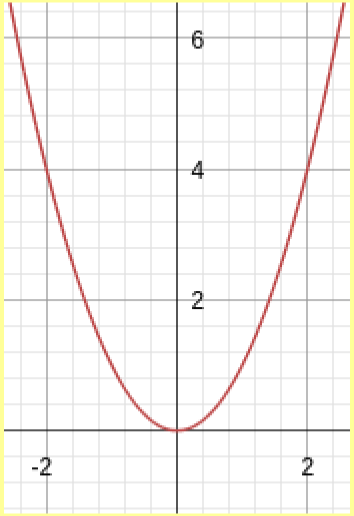
For every $f(x) = y$ in the original function, we will also have a $f^{-1}(y) = x$.

Remember that a function maps each element from the domain to one and only one element from the codomain.

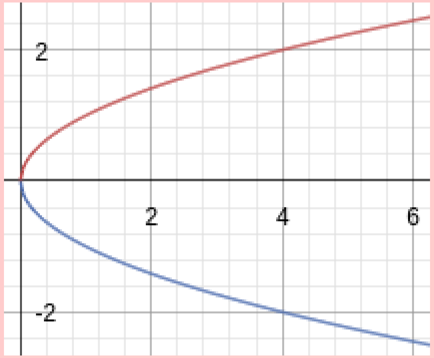
Notes

A function maps each element of the domain to one and only one element of the codomain.

For an example of a function whose inverse is *not a function*, we can look at f(x) = x^2, whose inverse is $f(x) = \pm\sqrt{x}$



$f(x) = x^2$



$f(x) = \pm\sqrt{x}$

Graphed with https://www.mathway.com/Graph

Notes

A function maps each element of the domain to one and only one element of the codomain.

1. Invertibility Property

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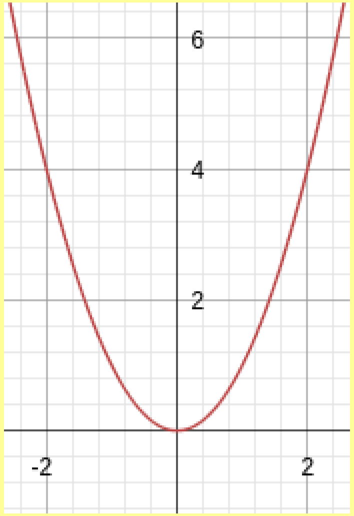
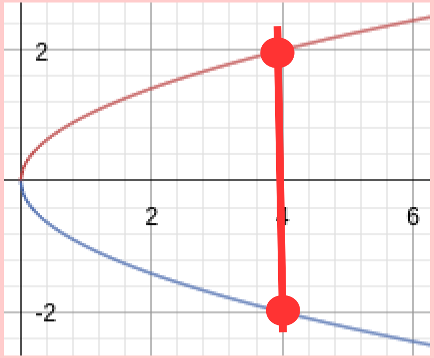


$f(x) = x^2$



$f(x) = \pm\sqrt{x}$

If we use the "vertical line test", we can see that there are some values of x that map to two y values.

Graphed with https://www.mathway.com/Graph

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A function maps <u>each element</u> of the domain to <u>one and only one</u> element of the codomain.

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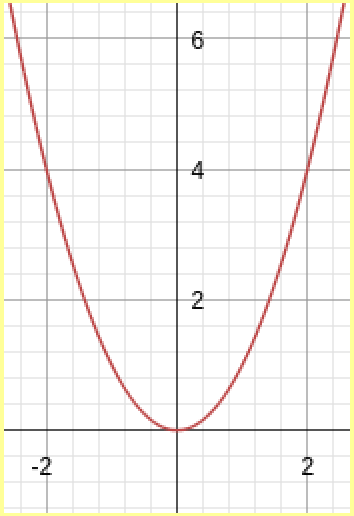
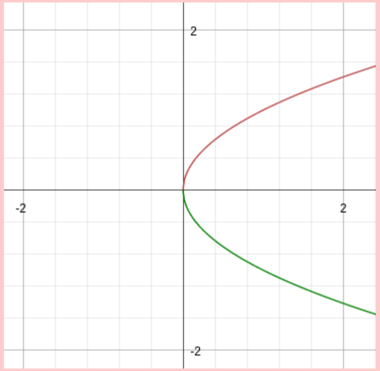


$f(x) = x^2$



$f(x) = \pm\sqrt{x}$

Additionally, not all elements of the domain (the x axis) have mappings to y values, which also makes this an invalid function.

Graphed with https://www.mathway.com/Graph

Notes

A function maps each element of the domain to one and only one element of the codomain.

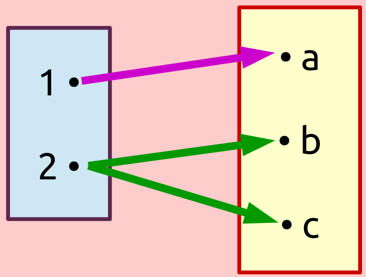
1. Invertibility Property

With our diagrams, we can visually check the two requirements for a valid function:

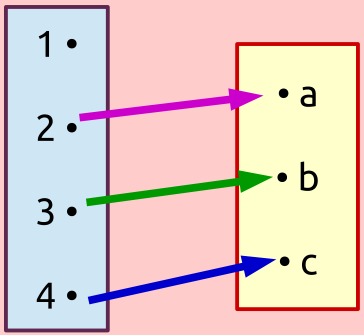
(1) All elements from domain map to something,

and

(2) No elements from the domain point to more than one item in the codomain.



Not a function



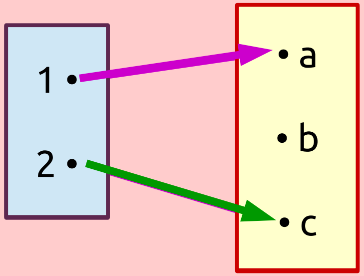
Not a function

Notes

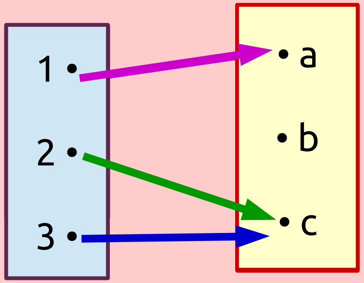
A function maps <u>each element</u> of the domain to <u>one and only one</u> element of the codomain.

1. Invertibility Property

To investigate whether a function is invertible, we think of those properties in reverse; if we inverted the domain and codomain and mappings, would the diagram be a valid function?



Function is valid,
Inverse is not.



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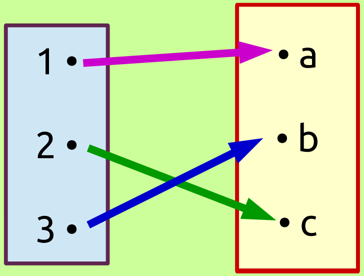

Function is valid,
Inverse is valid.

Notes

A function maps each element of the domain to one and only one element of the codomain.

6. Additional Notation

4. Additional Notation

1

For a function $f : A \to B$, writing out $(x, y) \in f$ means The input ***x*** *(from A)* is mapped to the output ***y*** *(from B)* in the rules of the function *f*.

The value x is from A, so

$$x \in A$$

The value y is from B, so

$$y \in B$$

And the mapping from x to y exists for the function, so

$$(x, y) \in f$$

Notes

Domain: Set of all possible inputs

Codomain: Set of all possible outputs.

$f : A \to B$
function f, with input from set A, & output from set B.

$R : A \to B$
A relation *R* has a subset of *A x B* as its rule.

5. Inverses

3. Inverses

We can also take the inverse of a function or a relation.

Given some relation $R : A \rightarrow B$,
the inverse is $R^{-1} : B \rightarrow A$

We also reverse the mappings, so a map from input x to output y (x, y)

would become (y, x) in the inverse.

Notes

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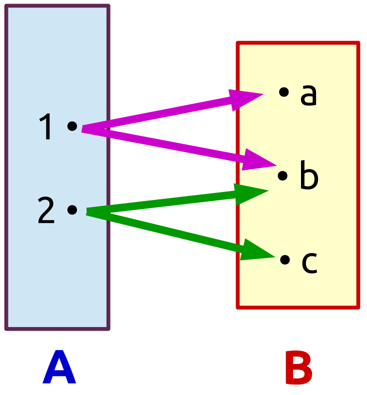
$f : A \rightarrow B$
function f, with input from set A, & output from set B.

$R : A \rightarrow B$
A relation R has a subset of $A \times B$ as its rule.

For example, say we have…

$R : A \rightarrow B$
A = { 1, 2 }　　　B = { a, b, c }
Rule: { (1, a), (1, b), (2, b), (2, c) }



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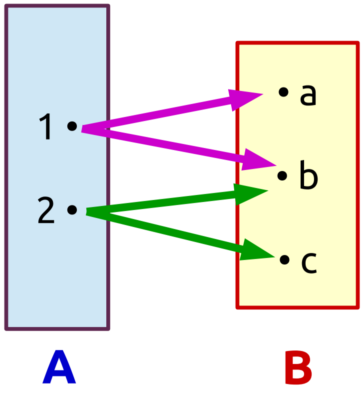
3. Inverses

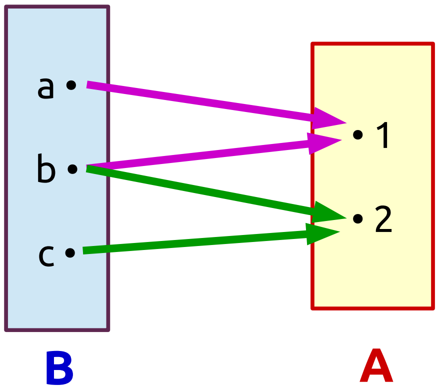
For example, say we have…

$R : A \to B$
A = { 1, 2 } B = { a, b, c }
Rule: { (1, a), (1, b), (2, b), (2, c) }

Its inverse will be...

$R^{-1} : B \to A$
A = { 1, 2 } B = { a, b, c }
Rule: { (a, 1), (b, 1), (b, 2), (c, 2) }





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$f : A \to B$
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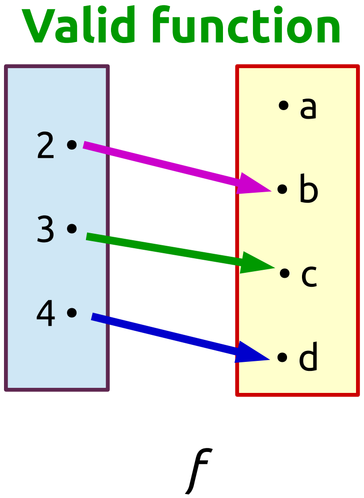
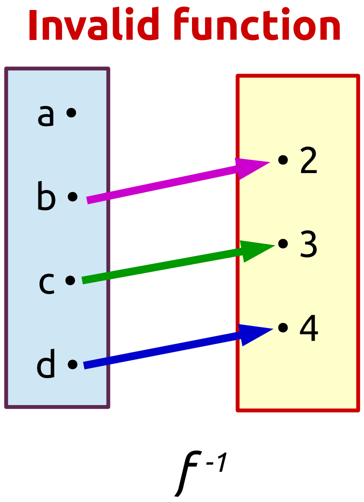
A relation *R* has a subset of *A x B* as its rule.

3. Inverses

Note that the inverse of a function will not always be a function...





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A relation R has a subset of $A \times B$ as its rule.

3. Inverses

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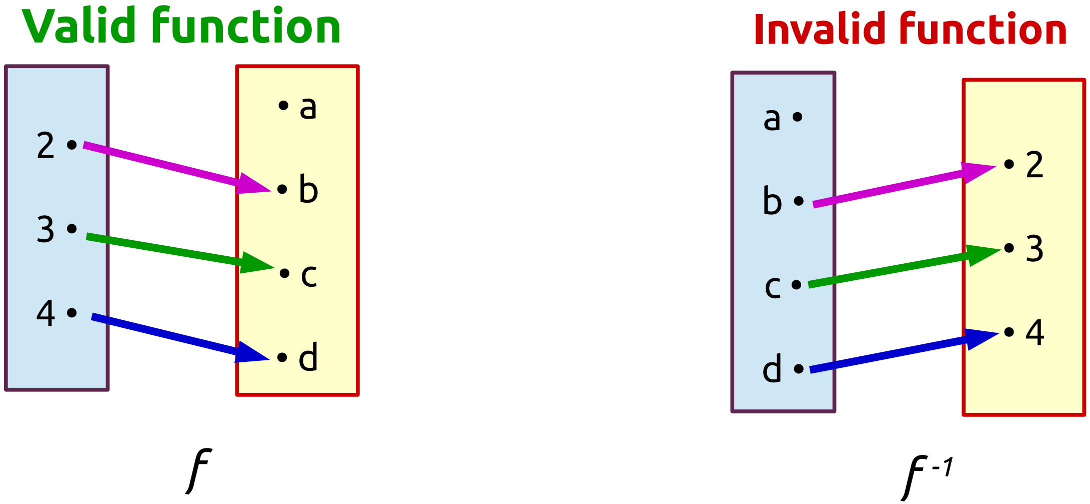

This is not a valid function because the input value "a" is not being mapped to anything.

Notes

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$f : A \rightarrow B$
function f, with input from set A, & output from set B.

$R : A \rightarrow B$
A relation R has a subset of $A \times B$ as its rule.

3. Inverses

Practice: Diagram the following relation and its inverse.

Relation: $R : A \to B$ $A = \{ 1, 2, 3 \}$ $B = \{ x, y, z \}$
Rule: $\{ (1, x), (1, y), (1, z), (2, x), (2, y), (3, z) \}$

Notes

Domain: Set of all possible inputs

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$f : A \to B$
function f, with input from set A, & output from set B.

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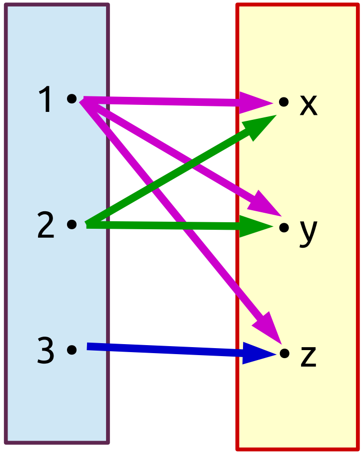
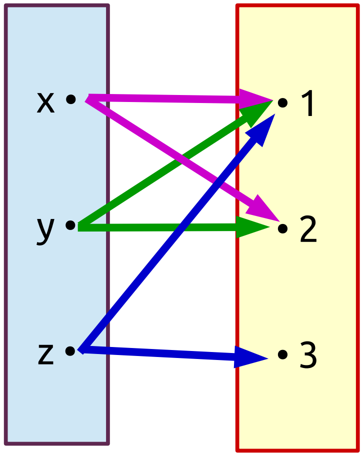
3. Inverses

Practice: Diagram the following relation and its inverse.

Relation: R : A → B   A = { 1, 2, 3 }   B = { x, y, z }
Rule: { (1, x), (1, y), (1, z), (2, x), (2, y), (3, z) }



R



R^{-1}

Relation: R^{-1} : B → A   A = { 1, 2, 3 }   B = { x, y, z }
Rule: { (x, 1), (y, 1), (z, 1), (x, 2), (y, 2), (z, 3) }

Notes

Domain: Set of all possible inputs

Codomain: Set of all possible outputs.

$f : A \to B$
function f, with input from set A, & output from set B.

$R : A \to B$
A relation *R* has a subset of *A x B* as its rule.

Conclusion

Make sure you understand these core concepts before continuing to the next topics.

Next time we will talk about properties of functions and properties of relations.