# Equivalence Relations

Written by Rachel Singh, last updated Oct 1, 2018

# About

Now we are looking at Equivalence Relations. This is when two items are considered equivalent in some way (defined by the rule given).

# Topics

1. Equivalence Relations

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Let's say we're playing a game by flipping 3 coins. For every Heads, you win $5. We could group the outcomes by the total amount of money won:

| $0 | $5 | $10 | $15 |
|---|---|---|---|
| TTT | TTH | HHT | HHH |
| | THT | HTH | |
| | HTT | THH | |

We can consider any set of outcomes under a single column as "equivalent", since they result in the same amount of money. So, for the purposes of our game, the "**TTH**" outcome is the "same" as the "**HTT**" outcome.

## Notes

**Reflexive:**
(a,a) ∈ R for all inputs

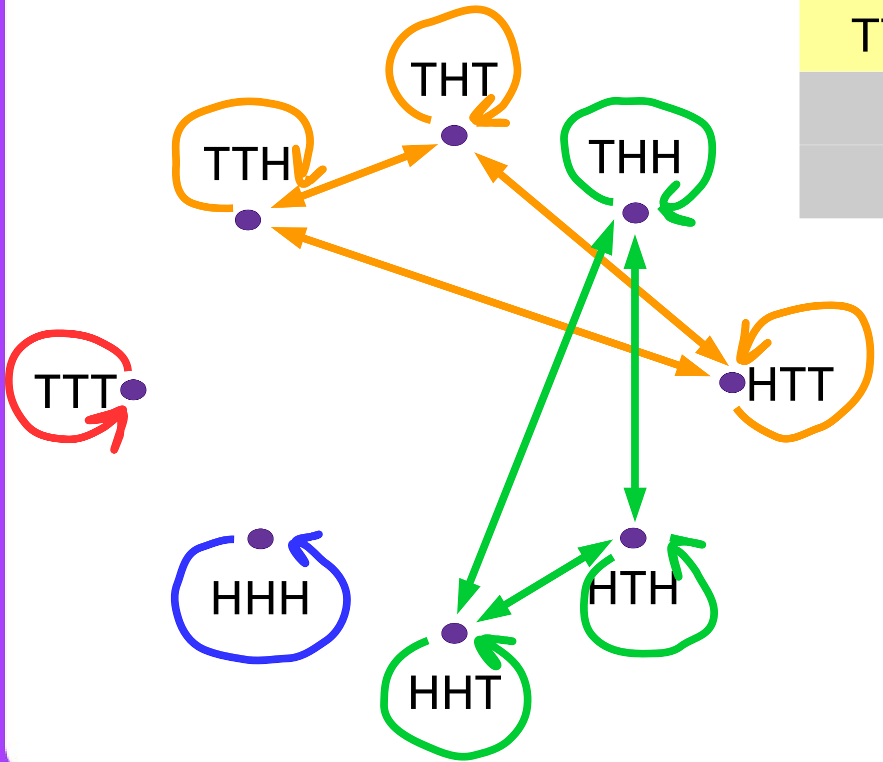
**Irreflexive:**
(a,a) ∉ R for all inputs

**Symmetric:**
(a,b) ∈ R and (b,a) ∈ R for all inputs where a ≠ b

**Antisymmetric:**
(a,b) ∈ R and (b,a) ∉ R for all inputs where a ≠ b

**Transitive:**
If (a,b) ∈ R and (b,c) ∈ R, then (a, c) ∈ R.

## We can model this with a graph as well…




| $0 | $5 | $10 | $15 |
|---|---|---|---|
| TTT | TTH | HHT | HHH |
| | THT | HTH | |
| | HTT | THH | |

**A = { TTT, TTH, THT, THH, HTT, HTH, HHT, HHH }**

**R = { (a, b) ∈ AxA : a and b have the same amount of H's}**

### Notes

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# 1. Equivalence Relations

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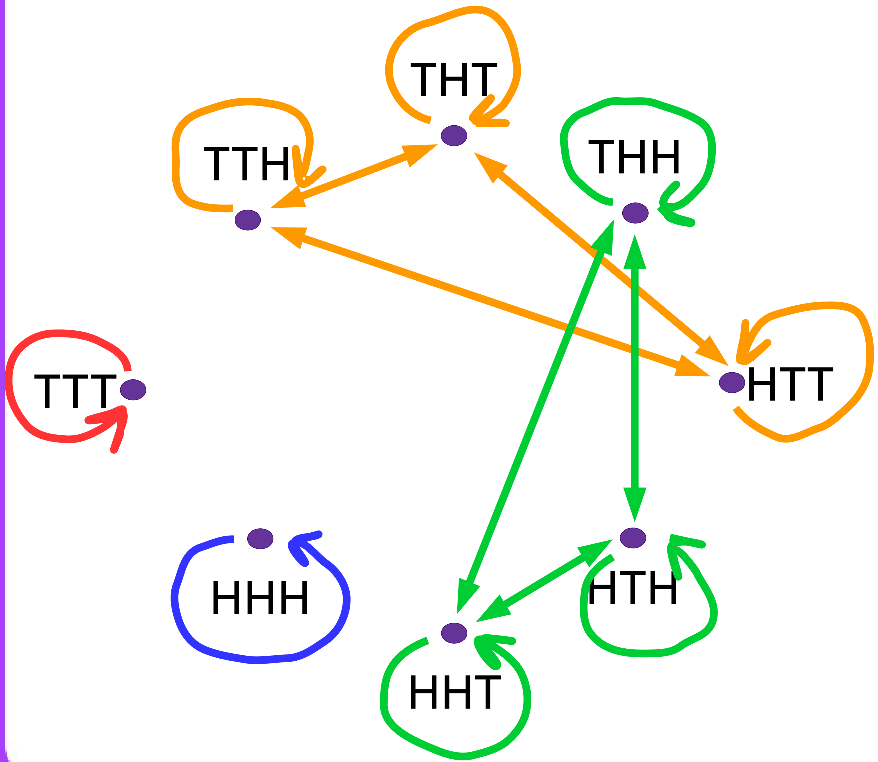


**Quick question, is this relation…**

- **Reflexive, irreflexive, or neither?**
- **Symmetric, antisymmetric, or neither?**
- **Transitive or intransitive?**

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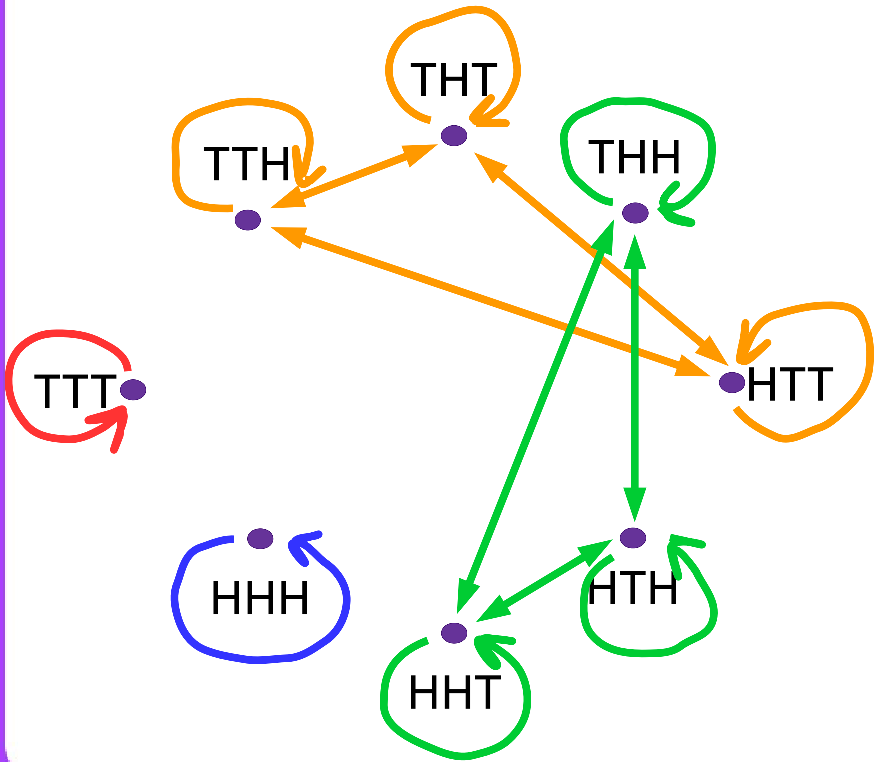


**Quick question, is this relation…**

- **Reflexive, irreflexive, or neither?**
- **Symmetric, antisymmetric, or neither?**
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**It is Reflexive, Symmetric, and Transitive.**

## Notes

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# 1. Equivalence Relations

We covered a lot about Relations and properties of relations last time, and those properties still apply here. Instead of a relationship where we *order* the items, we are finding *equivalence* instead.

Let's go over some more types of equivalence relations problems.

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If (a,b) ∈ R and (b,c) ∈ R, then (a, c) ∈ R.

# 1. Equivalence Relations

**Practice:** Draw the arrow diagram and decide whether the graph is (reflexive/irreflexive/neither), (symmetric/antisymmetric/neither), and (transitive/intransitive).

***R = { (A,B) ∈ ℘(S) x ℘(S) : A ⊆ B }***
**with *S = { 1, 2, 3 }***

*Hint: ℘(S) = { Ø, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }*

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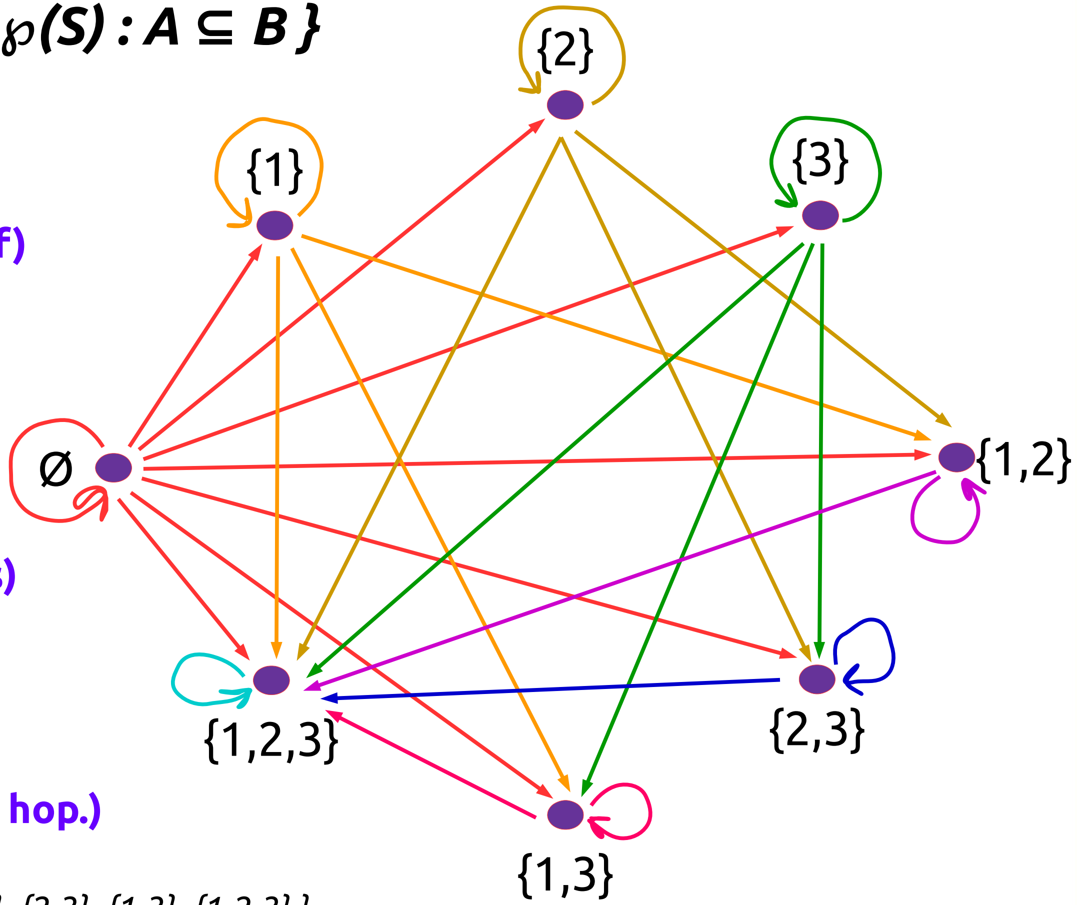
# 1. Equivalence Relations

***R* = { (*A*,*B*) ∈ ℘(*S*) x ℘(*S*) : *A* ⊆ *B* }**
**with *S* = { 1, 2, 3 }**

**It is Reflexive**
**(each node loops on itself)**




* **It is antisymmetric**
**(no arrow goes both ways)**

**It is transitive**
**(for all two-step hops,**
**there is a direct one-step hop.)**

*Hint: ℘(S) = { Ø, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }*

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# 1. Equivalence Relations

**Practice:** Draw the arrow diagram and decide whether the graph is (reflexive/irreflexive/neither), (symmetric/antisymmetric/neither), and (transitive/intransitive).

$$R = \{ (A,B) \in \wp(S) \times \wp(S) : A \cap B = \emptyset \} \qquad \text{with } S = \{ 1, 2, 3 \}$$

*Hint:* $\wp(S) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \}$

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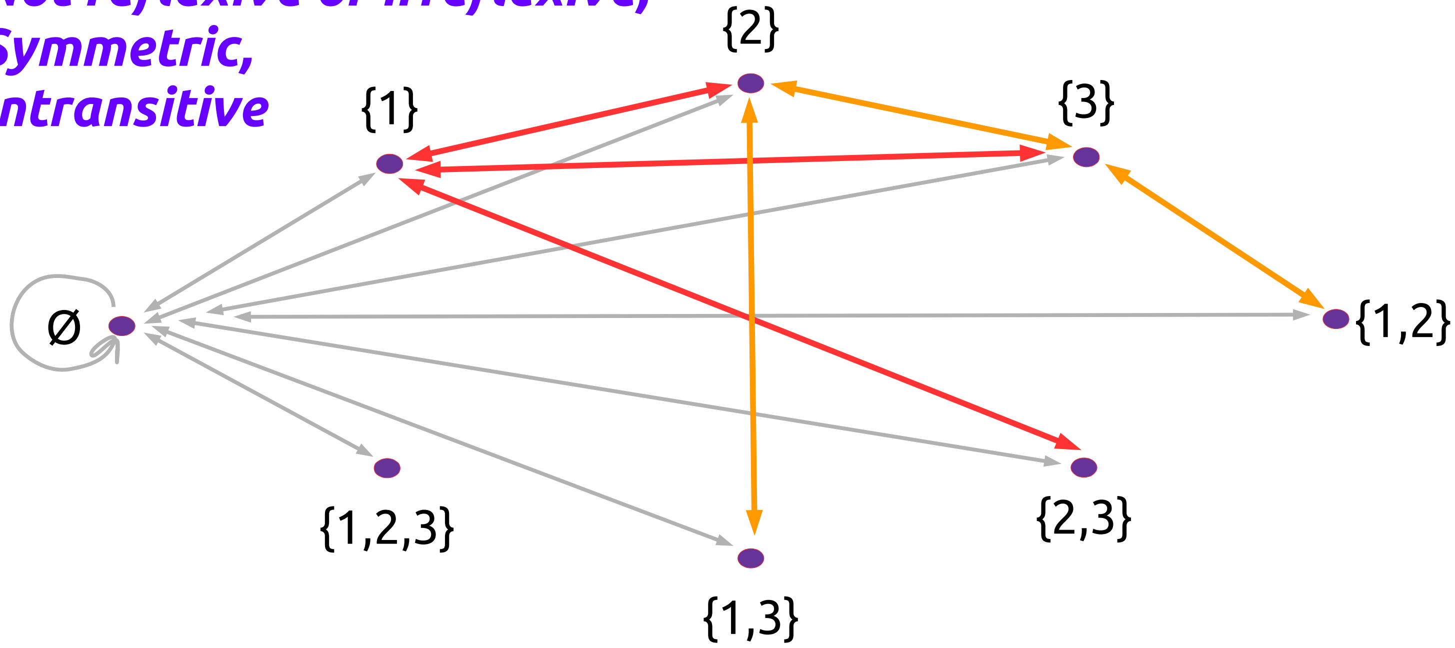
# 1. Equivalence Relations

**Practice:** Draw the arrow diagram and decide whether the graph is (reflexive/irreflexive/neither), (symmetric/antisymmetric/neither), and (transitive/intransitive).

***R = { (A,B) ∈ ℘(S) x ℘(S) : A∩B=Ø }*** ***with S = { 1, 2, 3 }***

***Not reflexive or irreflexive,***
***Symmetric,***
***Intransitive***

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***$R = \{ (A,B) \in \wp(S) \times \wp(S) : n(A)=n(B) \}$ with $S = \{ 1, 2, 3 \}$***

*Hint: $\wp(S) = \{ \varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \}$*

## Notes

**Reflexive:**
$(a,a) \in R$ for all inputs

**Irreflexive:**
$(a,a) \notin R$ for all inputs

**Symmetric:**
$(a,b) \in R$ and $(b,a) \in R$ for all inputs where $a \neq b$

**Antisymmetric:**
$(a,b) \in R$ and $(b,a) \notin R$ for all inputs where $a \neq b$

**Transitive:**
If $(a,b) \in R$ and $(b,c) \in R$, then $(a, c) \in R$.

# 1. Equivalence Relations

**Practice:** Draw the arrow diagram and decide whether the graph is (reflexive/irreflexive/neither), (symmetric/antisymmetric/neither), and (transitive/intransitive).

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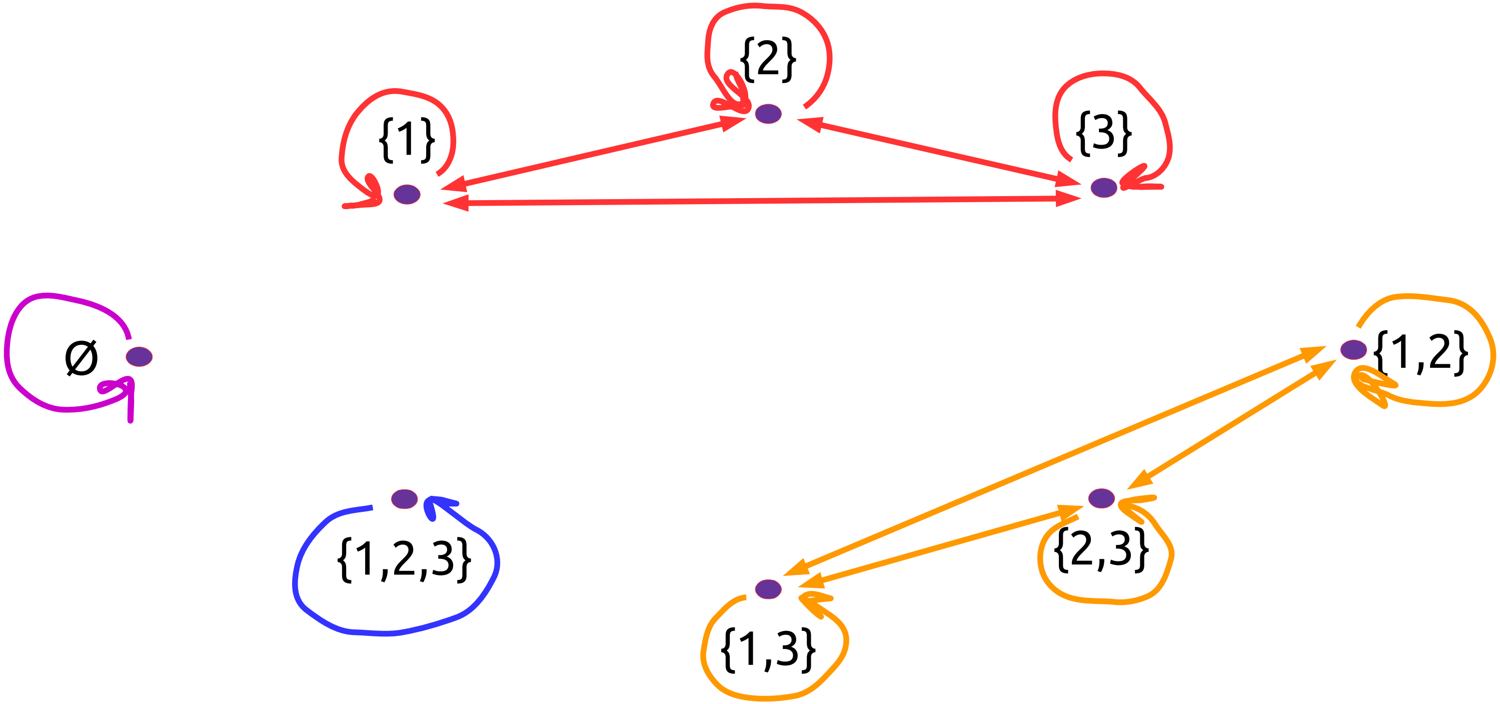


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# Conclusion

Make sure to practice diagramming ordered and equivalence relations, as well as identifying the different properties.