Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group**.

Names:

7.6 Graph Theory: Isomorphism and Planarity

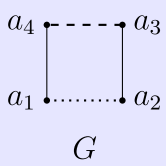
7.6.1 Isomorphism

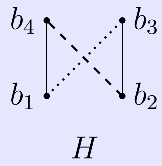
Isomorphism

Simple graphs G and H are called **isomorphic** if there is a one-to-one and onto function f from the nodes of G to the nodes of H such that $\{v, w\}$ is an edge of G if and only if $\{f(v), f(w)\}$ is an edge of H. The function f is called an isomorphism. Hence, an isomorphism is simply a **rule** associating nodes that preserves the edges joining the nodes. [a]

In other words, two graphs are isomorphic if they're essentially the same graph, with the vertices rearranged. Generally, you'll think of this as there being some function that maps a vertex from a graph G to another graph H.

Example:





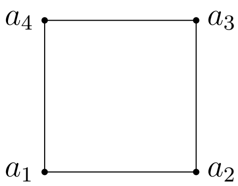
These are isomorphic... imagine taking a_2 and a_3 from graph G and physically flipping them with the edges still connected. In this case, our mapping is... $a_1 \mapsto b_1$ $a_2 \mapsto b_3$ $a_3 \mapsto b_2$ $a_4 \mapsto b_4$.

[a] Discrete Mathematics, Ensley and Crawley

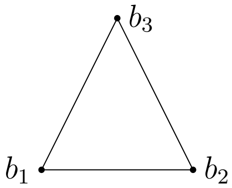
Question 1

Redraw the following graphs by moving the vertices around, but keeping the edges connected.

a.

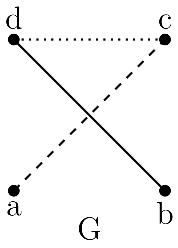


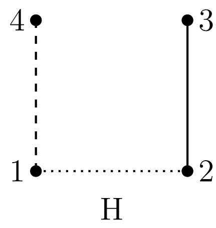
b.



Question 2

For the two graphs given, fill out a table of vertex mappings, and of edge mappings.





a. Vertex Map:

G	a	b	c	d
H				

b. Edge Map:

G	{a, c}	{c, d}	{d, b}
H			

Properties of isomorphic graphs

Two graphs that are isomorphic to one another must have...: [a]

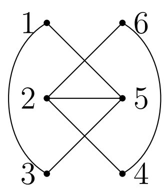
- The same number of nodes
- The same number of edges
- The same number of nodes of any given degree.
- The same number of cycles.
- The same number of cycles of any given size.

[a] Discrete Mathematics, Ensley and Crawley

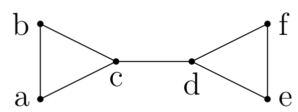
Question 3

Given the following two graphs...

G



H



a. Write out all edges for both graphs.

   G: {2, 5}

   H: {d, c}

b. For each edge from G, write out what edge in H corresponds to it.
   Example: {2,5} $\mapsto$ {d, c}

7.6.2 Planarity

Planarity

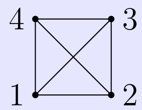
1. A simple, connected graph is called **planar** if there is a way to draw it (on a plane) so that no edges cross (i.e., they can only meet at a node). We will call "drawing" of a graph on a plane surface with no edge-crossings an **embedding** of the graph in the plane.

2. A graph is called **bipartite** if its set of nodes can be partitioned into two disjoint sets S_1 and S_2 so that every edge in the graph has one endpoint in S_1 and one endpoint in S_2.

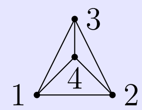
3. The **complete graph** on n nodes, denoted by K_n, is the simple graph with nodes $\{1, ..., n\}$ and an edge between every pair of distinct nodes.

4. The **complete bipartite graph** on n, m nodes, denoted by $K_{n,m}$, is the simple bipartite graph with nodes $S_1 = \{a_1, a_2, ..., a_n\}$ and $S_2 = \{b_1, b_2, ..., b_m\}$ and with edges connecting each node in S_1 to every node in S_2. [a]

Example: Let's redraw the graph K_4 so it has no overlapping edges.



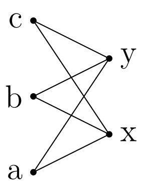
K_4



K_4 redrawn

[a] Discrete Mathematics, Ensley and Crawley

Question 4
Redraw the following graph, $K_{3,2}$, so that no edges are overlapping.



By Rachel Singh, last updated September 30, 2018

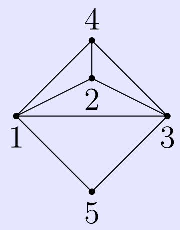
Faces

For a planar graph G embedded in the plane, a **face** of the graph is a region of the plane created by the drawing. Since the plane is an unbounded surface, every embedding of a finite planar graph will have exactly one **unbound** face. [a]

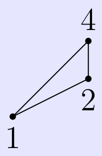
Unbound (external) face: Think of the external face as the "canvas" that all other faces are painted on to. Or, if you were viewing a silhouette of the drawing, you would only see the unbounded face - the sum of all the faces.

Example:

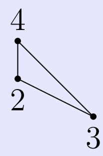
For the drawing, identify the faces by giving the cycle that creates each face, and highlight the unbounded face.



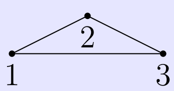
Faces:



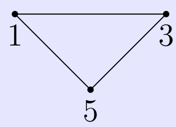
1, 2, 4, 1



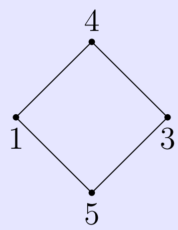
2, 3, 4, 2



1, 2, 3, 1



1, 3, 5, 1

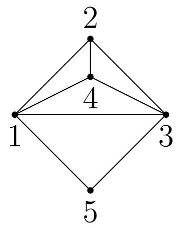


1, 4, 3, 5, 1
(unbounded)

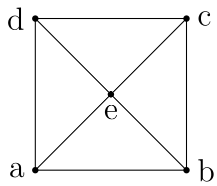
[a] Discrete Mathematics, Ensley and Crawley

Question 5
For both graphs, draw out each of its **faces**, then write out all the cycles bordering faces and identify the unbounded cycle.

a.



b.



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