Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group**.

Names:

Logic: Predicates and Quantifiers

Predicates

Predicates

In mathematical logic, a **predicate** is commonly understood to be a Boolean-valued function. [a]

We write a predicate as a function, such as $P(x)$, for example:

$P(x)$ is the predicate, "x is less than 2".

Once some value is plugged in for x, the result is a proposition - something either unambiguously **true** or **false**, but until we have some input for x, we don't know whether it is true or false.

$$P(0) = \text{true} \qquad P(2) = \text{false} \qquad P(10) = \text{false}$$

Additionally, predicates can also be combined with the logical operators AND $\wedge$ OR $\vee$ and NOT $\neg$.

[a] From https://en.wikipedia.org/wiki/Predicate_(mathematical_logic)

 By Rachel Singh, last updated October 1, 2018

Question 1
For the following predicates given, plug in **2, 23, -5, and 15** as inputs and write out whether the result is true or false.

a. $P(x)$ is the predicate "$x > 15$"

- $P(2) =$
- $P(23) =$
- $P(-5) =$
- $P(15) =$

b. $Q(x)$ is the predicate "$x \leq 15$"

- $P(2) =$
- $P(23) =$
- $P(-5) =$
- $P(15) =$

c. $R(x)$ is the predicate "$(x > 5) \wedge (x < 20)$"

- $P(2) =$
- $P(23) =$
- $P(-5) =$
- $P(15) =$

Domain

When we're working with predicates, we will also define the domain. The **domain** is the set of all possible inputs for our predicate. In other words, x must be chosen from the domain.

Question 2
For the following predicates and domains given, specify whether the predicate is true for **all members of the domain**, **some members of the domain**, or **no members of the domain.**

a. $P(x)$ is the predicate "$x > 15$", the domain is $\{10, 12, 14, 16, 18\}$.
☐ True for all ☐ True for some ☐ True for none

b. $Q(x)$ is the predicate "$x \leq 15$", the domain is $\{0, 1, 2, 3\}$.
☐ True for all ☐ True for some ☐ True for none

c. $R(x)$ is the predicate "$(x > 5) \wedge (x < 20)$", the domain is $\{0, 1, 2\}$.
☐ True for all ☐ True for some ☐ True for none

d. $S(x)$ is the predicate "$(x > 1) \wedge (x < 5)$", domain is $\{2, 3, 4\}$.
☐ True for all ☐ True for some ☐ True for none

Quantifiers

Quantifiers

Symbolically, we can specify that the input of our predicate, x, belongs in some domain set D with the notation: $x \in D$. This is read as, "x exists in the domain D."

Additionally, we can also specify whether a predicate is true **for all inputs x from the domain D** using the "for all" symbol $\forall$, or we can specify that the predicate is true **for *some* inputs x from the domain D** using the "there exists" symbol $\exists$.

Example: Rewrite the predicate symbolically.
$P(x)$ is "$x > 15$", the domain D is $\{16, 17, 18\}$. Here we can see that all inputs from the domain will result in the predicate evaluating to true, so we can write:

$\forall x \in D, P(x)$ ("For all x in D, x is greater than 15.")

- The symbol $\in$ ("in") indicates membership in a set.
- The symbol $\forall$ ("for all") means "for all", or "every".
- The symbol $\exists$ ("there exists") means "there is (at least one)", or "there exists (at least one)".
- The symbols $\forall$ and $\exists$ are called **quantifiers**. When used with predicates, the statement is called a **quantified predicate**.

Question 3
For the following predicates, rewrite the sentence symbolically, as in the example above. Use either $\forall$ or $\exists$, based on whether the predicate is true for the domain given.

Hint: If a predicate P(x) is false for all elements in the domain, you can phrase it as: "$\forall x \in D, \neg P(x)$".

a. $P(x)$ is the predicate "$x > 15$", the domain is $\{10, 12, 14, 16, 18\}$.

b. $Q(x)$ is the predicate "$x \leq 15$", the domain is $\{0, 1, 2, 3\}$.

c. $R(x)$ is the predicate "$(x > 5) \wedge (x < 20)$", the domain is $\{0, 1, 2\}$.

d. $S(x)$ is the predicate "$(x > 1) \wedge (x < 5)$", domain is $\{2, 3, 4\}$.

Question 4
For the following predicates, rewrite the sentence symbolically using the domain D = { 3, 4, 5, 10, 20, 25 }. Make sure to define your predicates (state that "$P(x)$ is the predicate...", and afterwards specify whether the quantified predicate is true or false. (If it states "there exists" but none exist, then the quantified predicate is false.)

a. There exists some m that is a member of D, such that $m \geq 3$.

b. There exists some element in D that is negative.

c. There is (at least one) k in the set D with the property that k^2 is also in the set D.

> **Hint**
> How can you specify the predicate, "k^2 is in the set D" symbolically?

Negating quantifiers

Negating quantifiers

For any predicates P and Q over a domain D,

- The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$.
- The negation of $\exists x \in D, P(x)$ is $\forall x \in D, \neg P(x)$.

When negating a predicate that uses an equal sign, the negation would be "not equals".

Example 1: Negate $\forall x \in \mathbb{Z}, x > 0$.

(For all integers x, x is greater than 0.)

1. $\neg(\forall x \in \mathbb{Z}, x > 0)$
2. $\equiv \neg(\forall x \in \mathbb{Z}), \neg(x > 0)$
3. $\equiv \exists x \in \mathbb{Z}, x \leq 0$

   (There exists some integer x such that x is less than or equal to than 0.)

In this case, the original statement "$\forall x \in \mathbb{Z}, x > 0$" was incorrect, and the negation "$\exists x \in \mathbb{Z}, x \leq 0$" is valid.

Question 5
Write the negation of each of these statements. Simplify as much as possible. Identify whether the original statement or the negation is true.

a. $\forall x \in \mathbb{Z}, \quad P(x)$.
$P(x)$ is "$2x$ is even".

b. $\exists x \in \mathbb{N}, \quad Q(x)$.
$Q(x)$ is "$x \cdot x < 0$".

c. $\forall x \in \mathbb{N}, \quad R(x)$.
$R(x)$ is "$x \in \mathbb{Z}$".

Question 6
Which elements of the set $D = \{2, 4, 6, 8, 10, 12\}$ make the **negation** of each of these predicates true? List the numbers from D that make the negation true.

a. $Q(n)$ is the predicate, "$n > 10$".

b. $R(n)$ is the predicate, "n is even".

c. $S(n)$ is the predicate, "$n^2 < 1$".

d. $T(n)$ is the predicate, "$n - 2$ is an element of D".