Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group**.

Names:

Logic: Nexted Quantifiers

Predicates

Nested Quantifiers

In some cases, we may want to work with multiple variables to build out a quantified statement. Whenever this occurs, we need to have quantifiers for each variable. That is, we need to specify "for all" or "there exists" for each variable, AND the domains for each.

Example: Write a quantified predicate for, "For every negative integer, there exists a positive integer, such that the sum of the two integers is positive."

- Two integers are specified, and one is positive and one is negative. Let's assign these variables the names x and y, and their domains will be $\mathbb{Z}^-$ and $\mathbb{Z}^+$.
- The proposition here is that adding the two variables will result in a positive result, so let's write this as "$x + y > 0$".
- It says "for every negative integer", this will be $\forall x \in \mathbb{Z}^-$.
- And "there exists some positive integer", so this will be $\exists y \in \mathbb{Z}^+$.

The Quantified Statement:

$$\forall x \in \mathbb{Z}^-, \exists y \in \mathbb{Z}^+, x + y > 0$$

Question 1
Translate the following English statement into a quantified predicate statement.

"For all integers x, there is some integer y, such that $x + y > 0$.

Use x as the first variable, and y as the second variable.

a. Give the quantifier and domain for x: *(For all... in...)*

b. Give the quantifier and domain for y:

c. Create a predicate with x and y as its inputs. It will be "$x + y$ is odd".

d. Put together the entire quantified statement.

Negating nested quantifiers

Proposition 1

For any predicates P and Q over a domain D,

- The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$.
- The negation of $\exists x \in D, P(x)$ is $\forall x \in D, \neg P(x)$.

When negating a predicate that uses an equal sign, the negation would be "not equals".

Example 1: Negate $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 0$.

1. $\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 0)$
2. $\equiv \quad \neg(\forall x \in \mathbb{Z}), \neg(\exists y \in \mathbb{Z}), \neg(x \cdot y = 0)$
3. $\equiv \quad \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y \neq 0$

Example 2: Given $N(n)$ is "n is negative" and $P(n)$ is "n is positive", negate $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, N(x) \wedge P(x) \rightarrow N(x \cdot y)$ (If x is negative and y is positive, then $x \cdot y$ is negative.)

1. $\neg(\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, N(x) \wedge P(x) \rightarrow N(x \cdot y))$
2. $\equiv \neg(\forall x \in \mathbb{Z}), \neg(\forall y \in \mathbb{Z}), \neg(N(x) \wedge P(x) \rightarrow N(x \cdot y))$
3. $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, N(x) \wedge P(x) \wedge \neg N(x \cdot y)$
   (There exists negative integer x and some positive integer y such that $x \cdot y$ is not negative.)

Question 2

Write the negation of each of these statements. Simplify as much as possible.

a. $\forall x \in \mathbb{Z}, \quad \exists y \in \mathbb{Z}, \quad 2x + y = 3$

b. $\exists x \in \mathbb{N}, \quad \forall y \in \mathbb{N}, \quad x \cdot y < x$

c. $\exists x \in \mathbb{Z}, \quad \exists y \in \mathbb{Z}, \quad (x + y = 13) \wedge (x \cdot y = 36)$

Hint:Negating propositions

From DeMorgan's laws,

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$