Implications
(aka Conditional Propositions)

Written by Rachel Singh, last updated Oct 1, 2018

About

In programming, we work with “if, then” statements a lot to decide how program flow branches.

In Discrete Math, we use a hypothesis and a conclusion to make a statement in the form,
“if the hypothesis is true, then the conclusion is true.”

This type of statement is called an implication. Implications and their negations have logic associated with them, which we can model with truth tables.

Topics

1. Implications
2. Truth Tables
   of implications
3. Negations
   of implications
4. Contrapositive,
   Converse, &
   Inverse

Implications

1. Implications

"If, then" statements are an integral part of programming, where we execute a set of instructions only *if* some condition evaluates to true.

```
if ( taller_than_48_inches )
{
    admitted_on_coaster = true;
}
```

With implications in Discrete Math, we identify two propositions; one is the **hypothesis** and one is the **conclusion**, and the implication is of the form,

"If the hypothesis is true, then the conclusion is true."

Notes

1. Implications

An implication is written in the form:

$$p \to q$$

Which can be read as, *"If p, then q"*, or *"p implies q"*.

In this case, **p** on the left of the arrow is the **hypothesis**, and **q** on the right side of the arrow is the **conclusion**.

In this case, we are using two **propositions**, p and q.

Notes

An implication is of the form $p \to q$

Left side of →: hypothesis

Right side of →: conclusion

1. Implications

So we could create an implication by specifying our propositions and building the implication…

- ***t*** is the proposition, "the guest is over 48 inches tall"
- ***r*** is the proposition, "the guest may ride the coaster"

t → r

"If the guest is over 48 inches tall,
then the guest may ride the coaster."

¬r → ¬t

"If the guest may NOT ride the coaster,
then the guest is NOT over 48 inches tall."

Notes

An implication is of the form
p → q

Left side of →: hypothesis

Right side of →: conclusion

1. Implications

Each side of the implication can be a formal proposition as well, combining multiple propositions together.

- ***a*** is the proposition, “Bob is 21 or over”,
- ***b*** is the proposition, “Bob can drink a beer”,
- ***s*** is the proposition, “Bob can drink a soda”.

$\neg a \rightarrow s$

“If Bob is NOT over 21, then bob can drink a soda.”

$a \rightarrow (b \lor s)$

“If Bob is over 21, then bob can drink a beer, or bob can drink a soda.”

Notes

An implication is of the form $p \rightarrow q$

Left side of →: hypothesis

Right side of →: conclusion

1. Implications

Each side of the implication can be a formal proposition as well, combining multiple propositions together.

- ***p*** is the proposition, "the printer has paper"
- ***o*** is the proposition, "the printer is out of order"
- ***d*** is the proposition, "anyone can print a document"

$$(p \land \neg o) \rightarrow d$$

*"If the printer has paper and the printer is **not** out of order, then anyone can print a document."*

Notes

An implication is of the form $p \rightarrow q$

Left side of →: hypothesis

Right side of →: conclusion

<u>Practice 1:</u> Write the following "if, then" statement symbolically as an implication.

"If <u>we can dance</u> and <u>your friends don't dance</u>, then <u>your friends are not my friends</u>"

Propositions:

w: We can dance **d: Your friends dance**

f: Your friends are my friends

Notes

An implication is of the form
p → q

Left side of →: hypothesis

Right side of →: conclusion

Practice 1: Write the following "if, then" statement symbolically as an implication.

"If we can dance and your friends don't dance, then your friends are not my friends"

Propositions:

w: We can dance **d: Your friends dance**

f: Your friends are my friends

$$(w \wedge \neg d) \rightarrow \neg f$$

Notes

An implication is of the form $p \rightarrow q$

Left side of →: hypothesis

Right side of →: conclusion

Truth Tables of Implications

2. Truth tables of implications

An implication, generally, is in the form:

"if ***the hypothesis is true***,
then ***the conclusion is true***"

Note that we already specify a hypothesis ("if…") and the conclusion ("then…").

An implication can be **true** or **false**, but this isn't the outcome (that's the conclusion) – so what does the "trueness" or "falseness" of an implication *mean?*

Notes

An implication is of the form
$p \to q$

Left side of →: hypothesis

Right side of →: conclusion

2. Truth tables of implications

The truth table of an implication is the following:

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

- Why is the implication only false if
  <u>the hypothesis is true</u> and the <u>conclusion is false</u>?

- Why is the implication true any time
  <u>the hypothesis is false</u>?

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

Again, if an implication is **false**, that doesn't mean the outcome is **false** - - the outcome is the conclusion!

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

Again, if an implication is **false**, that doesn't mean the outcome is **false** - - the outcome is the conclusion!

Think of it like a science experiment:
1) You create a hypothesis and the conclusion.

2) You test the hypothesis.

   a) **If the conclusion holds (conclusion = true):**
      Your experiment was a success!
      (implication = true)

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

Again, if an implication is **false**, that doesn't mean the outcome is **false** - - the outcome is the conclusion!

Think of it like a science experiment:

1) You create a hypothesis and the conclusion.

2) You test the hypothesis.

   a) **If the conclusion holds (conclusion = true):** Your experiment was a success! (implication = true)

   b) **If the conclusion fails (conclusion = false):** Your experiment was a failure! Time for a new hypothesis! (implication = false)

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

Again, if an implication is **false**, that doesn't mean the outcome is **false** - - the outcome is the conclusion!

Think of it like a science experiment:
1) You create a hypothesis and the conclusion.

2) You test the hypothesis.

   a) **If the conclusion holds (conclusion = true):**
   Your experiment was a success!
   (implication = true)

   b) **If the conclusion fails (conclusion = false):**
   Your experiment was a failure!
   Time for a new hypothesis! (implication = false)

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

But what about a "false hypothesis"? Why is the implication "true"??

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

Again, if an implication is **false**, that doesn't mean the outcome is **false** - - the outcome is the conclusion!

Think of it like a science experiment:
1) You create a hypothesis and the conclusion.

2) You get distracted by social media and forget to do your experiment.

   a) **If the conclusion holds (conclusion = true):** You… didn't even do the experiment. You didn't <u>prove</u> or <u>disprove</u> anything!

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

Again, if an implication is **false**, that doesn't mean the outcome is **false** - - the outcome is the conclusion!

Think of it like a science experiment:

1) You create a hypothesis and the conclusion.

2) You get distracted by social media and forget to do your experiment.

   a) **If the conclusion holds (conclusion = true):** You… didn't even do the experiment. You didn't prove or disprove anything!

   b) **If the conclusion fails (conclusion = false):** You… didn't even do the experiment. You didn't prove or disprove anything!

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

The conclusion here doesn't matter; your implication itself is invalid because you didn't even follow through on the hypothesis!

In this case, we consider the implication **vacuously true**, or **true by default**.

"Innocent until proven guilty"*

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

* you get the idea, though unfortunately this is not always the case in our justice system. See also: Weapons of Math Destruction, Black Box Thinking

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

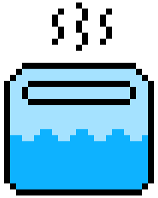
Here’s another example:

Let’s say that we’ve come up with a scientific hypothesis that we want to test…

“If you watch a pot of water, then the water will never boil.”

The hypothesis is, “You watch a pot of water”, and the conclusion is “the water will never boil”.

We are going to perform an experiment to test this hypothesis.



Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

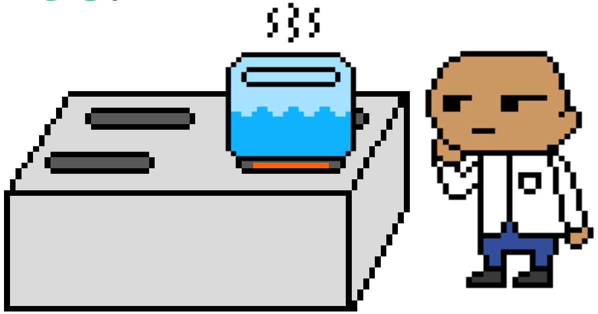
hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

To test the hypothesis, we heat up a pot of water and watch it.

If the water, indeed, never boils, then the **hypothesis** and the **conclusion** were both true, and the entire **implication is true**.



Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

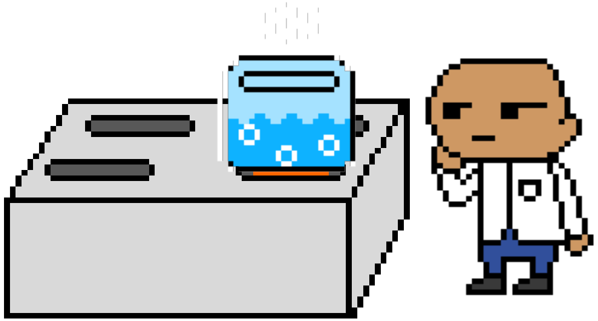
hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

To test the hypothesis, we heat up a pot of water and watch it.

If the water begins boiling, then while our **hypothesis** was true, the **conclusion is false**, so the **entire implication is false**.



Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

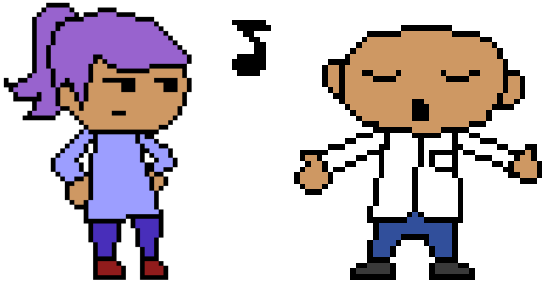
hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

However, let's say we ***don't*** watch the pot of water – the hypothesis is ***false***.

What does it mean if we're not watching the pot, and the conclusion is **true** – the pot never boils?



Notes

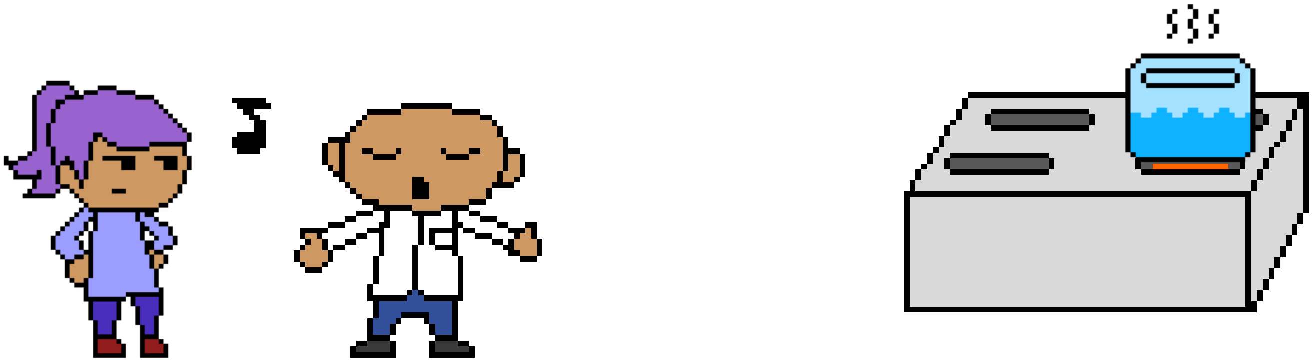
If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

"If you watch a pot of water, then the water will never boil."

Well, we haven't proven our implication, and we haven't *disproven it, either*. The logical result of the implication, "if you watch a pot of water, then the water will never boil", is going to be ***true***.



Notes

If [hypothesis]
Then [conclusion]

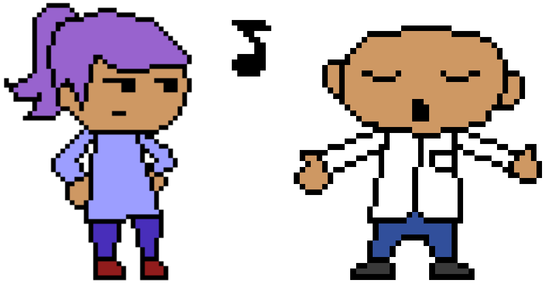
hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

Maybe that seems weird logically, but think of it as, "the implication still holds – we haven't disproven it. This is because we didn't even do the hypothesis *correctly.*"



Notes

If [hypothesis]
Then [conclusion]

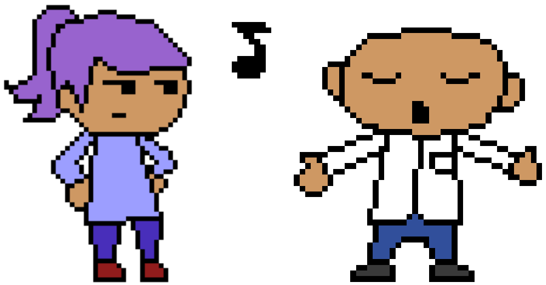
hypothesis → conclusion

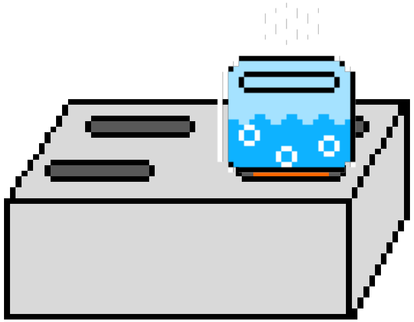
hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

Similarly, if we're not watching the pot of water (**hypothesis is false**), but it begins boiling (**conclusion is false**), the result of this **implication is *also true*.**





Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

2. Truth tables of implications

We can only say our implication is **false** if we actually disprove it by making sure the **hypothesis is true** and the **conclusion turns out to be false**.

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

Negations of Implications

3. Negations of implications

Previously, we saw that if a proposition was **false**, we could also say that the *negation* of that statement is **true**.

We can also negate our implications.

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

Let's say the doctor tells you,

"If you exercise more, you will sleep better"

p is the proposition, "you exercise more"
q is the proposition, "you will sleep better"

The only way the doctor's implication is ***false*** is if you exercise more, but ***don't*** sleep better.

If you don't exercise, you can't judge the sleep outcome, whether you end up sleeping better or not!

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

"If you exercise more, you will sleep better"

p is the proposition, "you exercise more"
q is the proposition, "you will sleep better"

If the implication is **false**, we can figure out the negation simply by using the truth table:

p	q	p → q	¬(p → q)
True	True	True	False
True	False	False	True
False	True	True	False
False	False	True	False

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

We've seen the negations of AND and OR statements, and how we can "expand" the expressions…

AND

p	q	p ∧ q	¬(p ∧ q)	¬p ∨ ¬q
True	True	True	False	False
True	False	False	True	True
False	True	False	True	True
False	False	False	True	True

$\neg(p \land q) \equiv \neg p \lor \neg q$

OR

p	q	p ∨ q	¬(p ∨ q)	¬p ∧ ¬q
True	True	True	False	False
True	False	True	False	False
False	True	True	False	False
False	False	False	True	True

$\neg(p \lor q) \equiv \neg p \land \neg q$

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

In generalized terms (looking at just "p" and "q"), how could we build an expression equivalent to

$\neg(p \rightarrow q)$?

p	q	$\neg(p \rightarrow q)$
True	True	False
True	False	True
False	True	False
False	False	False

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

It’s not ¬p → ¬q…

p	q	¬(p → q)	¬p → ¬q ?
True	True	False	True
True	False	True	True
False	True	False	False
False	False	False	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

It’s not ¬p → ¬q…
It’s not p → ¬q…

p	q	¬(p → q)	¬p → ¬q ?	p → ¬q ?
True	True	False	True	False
True	False	True	True	True
False	True	False	False	True
False	False	False	True	True

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

Well… if we look at the states of **p** and **q**, we can see that **p** will be true, AND **q** will be false…

p	q	¬(p → q)
True	True	False
True	False	True
False	True	False
False	False	False

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

Well… if we look at the states of **p** and **q**, we can see that **p** will be true, AND **q** will be false…

p	q	¬(p → q)	p ∧ ¬q
True	True	False	False
True	False	True	True
False	True	False	False
False	False	False	False

So, **p ∧ ¬q** is logically equivalent to **¬(p → q)**.

The negation of **¬(p → q)** is NOT an implication.

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

¬(p → q)
≡ p ∧ ¬q

3. Negations of implications

$\neg(p \to q) \equiv p \wedge \neg q$

p	q	p → q	¬(p → q)	p ∧ ¬q
True	True	True	False	False
True	False	False	True	True
False	True	True	False	False
False	False	True	False	False

The truth table of p → q, and the negation of an implication are two topics that are easy to forget and mix up! Make sure to take note of this!

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \to q)$
$\equiv p \wedge \neg q$

3. Negations of implications

Practice 2: Find the negation of the following implications.

1. $p \rightarrow (q \wedge r)$

2. $(p \vee q) \rightarrow r$

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \rightarrow q)$
$\equiv p \wedge \neg q$

3. Negations of implications

Practice 2: Find the negation of the following implications.

1. $p \to (q \land r)$
   $\equiv \neg(p \to (q \land r))$
   $\equiv p \land \neg(q \land r)$
   $\equiv p \land (\neg q \lor \neg r)$

2. $(p \lor q) \to r$
   $\equiv \neg((p \lor q) \to r)$
   $\equiv \neg(p \lor q) \land \neg r$
   $\equiv \neg p \land \neg q \land \neg r$

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \to q)$
$\equiv p \land \neg q$

3. Negations of implications

Practice 3: Give the negation of the following, both symbolically and in English.

The doctor tells you,
"If you eat healthy and exercise then you will lose weight."

h: you eat healthy, **e**: you exercise, **w**: you lose weight

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \to q)$
$\equiv p \wedge \neg q$

3. Negations of implications

Practice 3: Give the negation of the following, both symbolically and in English.

The doctor tells you,
"If you eat healthy and exercise then you will lose weight."

h: you eat healthy, **e**: you exercise, **w**: you lose weight

- Original: $(h \wedge e) \rightarrow w$
- Negation: $\neg[(h \wedge e) \rightarrow w] \equiv (h \wedge e) \wedge \neg w$
- English: ***You eat healthy and you exercise, and you do not lose weight.****

* You are beautiful as you are, and while you should strive to eat well and avoid a sedentary lifestyle, losing weight and trying to meet our society's narrow definition of "attractiveness" shouldn't be the end goal.

Notes

If [hypothesis]
Then [conclusion]

hypothesis → conclusion

hyp	con	hyp → con
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

Converse, Inverse, and Contrapositives

4. Converse, Inverse, and Contrapositives

Sometimes, it can be helpful to *reframe* an implication in other ways.

If we look at an **implication** and its **contrapositive**, they will be logically equivalent to each other.

And the **inverse** of an implication, and the **converse** of the same implication, will also be logically equivalent.

Notes

Given $p \rightarrow q$,

Converse:
$\mathbf{q \rightarrow p}$

Inverse:
$\mathbf{\neg p \rightarrow \neg q}$

Contrapositive:
$\mathbf{\neg q \rightarrow \neg p}$

4. Converse, Inverse, and Contrapositives

Given some implication:

$$p \to q$$

The **converse** is of the form:

$$q \to p$$

(This is not equivalent to p → q,
but I've included both in the truth table for comparison:)

p	q	p → q	q → p
True	True	True	True
True	False	False	True
False	True	True	False
False	False	True	True

Notes

Given p → q,

Converse:
q → p

Inverse:
¬p → ¬q

Contrapositive:
¬q → ¬p

4. Converse, Inverse, and Contrapositives

So for the implication,

"If you are a rich man,
then you don't have to work hard"

The converse is:

"If you don't have to work hard,
then you are a rich man"

Notes

Given p → q,

Converse:
q → p

Inverse:
¬p → ¬q

Contrapositive:
¬q → ¬p

4. Converse, Inverse, and Contrapositives

Given some implication:

$$p \to q$$

The **inverse** is of the form:

$$\neg p \to \neg q$$

(The inverse and the converse are logically equivalent):

p	q	p → q	q → p	¬p → ¬q
True	True	True	True	True
True	False	False	True	True
False	True	True	False	False
False	False	True	True	True

Notes

Given p → q,

Converse:
q → p

Inverse:
¬p → ¬q

Contrapositive:
¬q → ¬p

4. Converse, Inverse, and Contrapositives

So for the implication,

"If you collect 100 coins,
then you get an extra life"

The inverse is:

"If you don't collect 100 coins,
then you don't get an extra life"

Notes

Given $p \to q$,

Converse:
$q \to p$

Inverse:
$\neg p \to \neg q$

Contrapositive:
$\neg q \to \neg p$

4. Converse, Inverse, and Contrapositives

Given some implication:

$$p \rightarrow q$$

The **contrapositive** is of the form:

$$\neg q \rightarrow \neg p$$

p	q	p → q	¬p	¬q	¬q → ¬p
True	True	True	False	False	True
True	False	False	False	True	False
False	True	True	True	False	True
False	False	True	True	True	True

Notes

Given p → q,

Converse:
q → p

Inverse:
¬p → ¬q

Contrapositive:
¬q → ¬p

4. Converse, Inverse, and Contrapositives

So for the implication,

"If you like SHINee,
then you like Korean Pop"

The contrapositive is:

"If you don't like Korean Pop,
then you don't like SHINee"

Notes

Given $p \to q$,

Converse:
$q \to p$

Inverse:
$\neg p \to \neg q$

Contrapositive:
$\neg q \to \neg p$

4. Converse, Inverse, and Contrapositives

Practice 3: Find the contrapositive, converse, and inverse of the following implication in English.

$$p \rightarrow q$$

"If pineapple is good, then pineapple belongs on pizza"

Where **p** is "pineapple is good" and **q** is "pineapple belongs on pizza"

1. **Contrapositive:** $\neg q \rightarrow \neg p$
2. **Converse:** $q \rightarrow p$
3. **Inverse:** $\neg p \rightarrow \neg q$

Notes

Given $p \rightarrow q$,

Converse:
$q \rightarrow p$

Inverse:
$\neg p \rightarrow \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

4. Converse, Inverse, and Contrapositives

Practice 3: Find the contrapositive, converse, and inverse of the following implication in English.

$$p \rightarrow q$$

"If pineapple is good, then pineapple belongs on pizza"

Where **p** is "pineapple is good" and **q** is "pineapple belongs on pizza"

1. **Contrapositive:** $\neg q \rightarrow \neg p$
   "If pineapple doesn't belong on pizza, then pineapple isn't good."
2. **Converse:** $q \rightarrow p$
   "If pineapple belongs on pizza, then pineapple is good."
3. **Inverse:** $\neg p \rightarrow \neg q$
   "If pineapple is not good, then pineapple does not belong on pizza."

Notes

Given $p \rightarrow q$,

Converse:
$q \rightarrow p$

Inverse:
$\neg p \rightarrow \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

Conclusion

Remember the truth tables for implications and the negation of an implication, as these two things tend to trip up students!

By covering propositions, predicates, quantified statements, and implications, we have covered the foundation of logic. Your brain should now be wired like a robot. Please be a benevolent robot.

But overall, this will help you understand the way we use logic in our computer programs to define the program's flow.