

PREDICATES AND QUANTIFIERS

ABOUT

In this lecture, we are working with predicates. Predicates behave similarly to our propositions from the last lecture, but this time they receive input. This input affects whether the predicate will be true or false.

We are also going to work with the quantifiers, “For All” and “There Exists” along with our predicates, to show that some predicate “ $P(x)$ ” is *always true* or *sometimes true* for any given input passed in as x .

TOPICS

1. Predicates

4. Negations

2. Quantifiers
and domain

5. Common numerical
sets

3. Counter-examples

6. Nested Quantifiers

PREDICATES

1. PREDICATES

Predicates are similar to propositions, except that predicates take an **input**, and the result will be a proposition that is either **true** or **false** based on that input.

If we make a statement like, “x is positive”, then whether the statement is true or not depends on what x is: **True** for 10, **false** for -5.

Notes

Predicate P(x):
A statement that takes in some input x, and results in either a **true** or **false** proposition.

1. PREDICATES

While propositions are written with lowercase letters (usually p and q),

Predicates are written with uppercase letters and their input variable, like $P(x)$ and $Q(x)$.

Example:



$P(x)$ is the predicate, "x is a cool dude."

Notes

Predicate P(x):
A statement that takes in some input x , and results in either a **true** or **false** proposition.

1. PREDICATES

In programming terms, think of propositions like our **boolean variables**, and predicates like defining a **function** that will return either **true** or **false**.

Proposition

```
bool pizzaIsLate = false;
```

Predicate

```
bool PizzaIsLate( int minutesElapsed )  
{  
    return ( minutesElapsed > 45 );  
}
```

Predicate returns a proposition

```
bool pizzaIsLate = PizzaIsLate( 60 );
```

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Predicate returns a proposition

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bool pizzaIsLate = PizzaIsLate( 60 );
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```
if ( pizzaIsLate )  
{  
    Output( "WHERE" );  
    Output( "IS" );  
    Output( "MY" );  
    Output( "PIZZA?!" );  
}
```

Notes

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if ( pizzaIsLate )  
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    Output( "MY" );  
    Output( "PIZZA?!" );  
}
```

```
WHERE  
IS  
MY  
PIZZA?!
```

Notes

Predicate P(x):
A statement that takes in some input x, and results in either a **true** or **false** proposition.

1. PREDICATES

We could translate this from programming terms:

```
bool PizzaIsLate( int minutesElapsed )  
{  
    return ( minutesElapsed > 45 );  
}
```

To Discrete Math terms:

$P(x)$ is the predicate, “ x is greater than 45”.



Notes

Predicate $P(x)$:
A statement that takes in some input x , and results in either a **true** or **false** proposition.

1. PREDICATES

We can also use our logic operators \wedge , \vee , and \neg with predicates.

$P(x)$ is the predicate, "x is even"
 $Q(x)$ is the predicate, "x ends in 0"

$P(x) \wedge Q(x)$: *x is even and x ends in 0*

$P(x) \vee Q(x)$: *x is even or x ends in 0*

$\neg P(x)$: *x is not even*

Are these statements **true** or **false**?
We don't know until we specify x !

Notes

Predicate P(x):
A statement that takes in some input x , and results in either a **true** or **false** proposition.

Logic Operators:
 \wedge And
 \vee Or
 \neg Not

1. PREDICATES

Practice 1:

Given the following predicate:
 $P(x)$ is the predicate, "x is prime"
Decide whether each of the following results to *true* or *false*.

x is 3

x is 9

Notes

Predicate P(x):
A statement that takes in some input x , and results in either a **true** or **false** proposition.

Logic Operators:

\wedge And

\vee Or

\neg Not

1. PREDICATES

Practice 1:

Given the following predicate:
 $P(x)$ is the predicate, "x is prime"
Decide whether each of the following
results to *true* or *false*.

x is 3

P(3) is true

x is 9

P(9) is false

Notes

Predicate P(x):

A statement that takes in some input x , and results in either a **true** or **false** proposition.

Logic Operators:

\wedge And

\vee Or

\neg Not

DOMAIN

QUANTIFIERS AND DOMAIN

2. QUANTIFIERS AND DOMAIN

Sometimes, no matter what you plug into a predicate, it will *always* come out to the same result (either always **true** or always **false**)

If this is the case, we can use a symbol that means “**for all x** ”, to specify that, “*For all input values of x* ”, the predicate will evaluate to the same result.

Notes

2. QUANTIFIERS AND DOMAIN

On the other hand, if a predicate may be **true** for some values of x , and **false** for other values of x , we can use a symbol to say, "*There exists some input value x* " such that the the predicate will give some result.

Notes

2. QUANTIFIERS AND DOMAIN

“For all” is denoted by \forall

And

“There exists” is denoted by \exists

So $\forall x$ would be read as, “For all x...”,

and $\exists x$ would be read as, “There exists some x...”

Notes

For all: \forall

There exists: \exists

2. QUANTIFIERS AND DOMAIN

Further, when we're making a quantified statement, we also need to specify what the **domain** is.

"In mathematics, and more specifically in naive set theory, the domain of definition (or simply the domain) of a function is the set of "input" or argument values for which the function is defined."

(From Wikipedia https://en.wikipedia.org/wiki/Domain_of_a_function)

In other words, the **domain** is a **set** of all possible input values we can use.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

So, putting this all together, let's say we have a set **D** with a set of numbers...

$$D = \{ 2, 4, 6, 8, 10 \}$$

When we choose some x value to plug into our predicate, x will be selected from the domain **D**.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

$$D = \{2, 4, 6, 8, 10\}$$

In this case, we can see that all elements of the domain D are even numbers, so we could say...

$E(x)$ is the predicate, "x is even"

\forall

For all

x

elements x

\in

in the set

$D,$

$D,$

$E(x)$

The predicate
 $E(x)$ is true.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

$$D = \{2, 4, 6, 8, 10\}$$

E(x) is the predicate, "x is even"

$$\forall x \in D, E(x)$$

So for our entire **quantified statement**, we need:

- 1) The quantifier (\forall or \exists)
- 2) Defining the input variable (x) as a member of the domain D (or some other domain)
- 3) The statement that will be true for all input values of x .

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Let's say we changed this around so that not all elements of D were even:

$$D = \{1, 2, 4, 6, 8, 10\}$$

$E(x)$ is the predicate, "x is even"

In this case, **$\forall x \in D, E(x)$** is **not** valid here.

We could say that,
there exists an x in D , such that x is not even.

Or, symbolically, **$\exists x \in D, \neg E(x)$**

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

$$D = \{ 1, 2, 4, 6, 8, 10 \}$$

E(x) is the predicate, "x is even"

So what can we say about this domain and this predicate?

- $\exists x \in D, E(x)$ There exists some element x in D such that x is even.
- $\exists x \in D, \neg E(x)$ There exists some element x in D such that x is not even.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

$$D = \{ 1, 2, 4, 6, 8, 10 \}$$

E(x) is the predicate, "x is even"

And the following statements would be false:

- **$\forall x \in D, E(x)$** For all elements x in D , x is even.
- **$\forall x \in D, \neg E(x)$** For all elements x in D , x is not even.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

So in order to state $\forall x \in D, P(x)$
it must be true for *all input variables*.

And in order to state $\exists x \in D, P(x)$
it must be true for *at least one input variable*.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 2:

Translate the following into English, and specify whether the quantified statement is true or false.

$\forall x \in D, P(x),$

$P(x)$ is the predicate, "x is negative".

$D = \{-5, -10, -15, -20\}$

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 2:

Translate the following into English, and specify whether the quantified statement is true or false.

$\forall x \in D, P(x),$

$P(x)$ is the predicate, "x is negative".

$D = \{-5, -10, -15, -20\}$

For all elements x in D, x is negative.

For this domain D, this quantified statement is true.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 3: Translate the following into English.

$\exists d \in D, G(d),$

$G(d)$ is the predicate, “ d is a good boy”

D is the set of all dogs.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 3: Translate the following into English.

$\exists d \in D, G(d),$

$G(d)$ is the predicate, “ d is a good boy”

D is the set of all dogs.

**There exists some dog in the set of all dogs,
such that dog d is a good boy.**

(One may argue that $\forall d \in D, G(d)$, however)

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 3:

Translate the following into a quantified statement. You must define the domain and the predicate.

All numbers between 1 and 10 are positive.

Hints:

- Is this \forall or \exists ?
- What is the domain?
- Define the predicate.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 3:

Translate the following into a quantified statement. You must define the domain and the predicate.

All numbers between 1 and 10 are positive.

$\forall x \in D, P(x)$

Where $D = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$
And $P(x)$ is the predicate, " $x > 0$ ".

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 4:

Translate the following into a quantified statement. You must define the predicate.

Given the domain $D = \{ 1, 1, 2, 3, 5, 8 \}$, some numbers are even.

Hints:

- For your predicate, just write **“is even”** in English. We have not yet covered how to specify that an integer is even mathematically; that’s for another lesson.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

2. QUANTIFIERS AND DOMAIN

Practice 4:

Translate the following into a quantified statement. You must define the predicate.

Given the domain $D = \{ 1, 1, 2, 3, 5, 8 \}$, some numbers are even.

$\exists x \in D, E(x)$

Where $E(x)$ is the predicate, “ x is even”.*

* A common error starting out is for students to try to write “ x is even” mathematically. They usually write it wrong, so don't! We will cover definitions of *even* and *odd* numbers in another section.

Notes

For all: \forall

There exists: \exists

Domain:

Set of input values

$x \in D$

x is an element of D

COUNTEREXAMPLES

3. COUNTER-EXAMPLES

If we have a quantified statement of the form:

$$\forall x \in D, P(x)$$

If we can find *at least one* x that makes $P(x)$ false, then we can disprove the entire quantified statement with this **counter-example**.

We can only use a counter-example to disprove a statement that specifies “for all”, but we only need one example to do so.

Notes

You can use a single counter-example to disprove a quantified statement that uses “for all”, if you can find an example that makes the predicate false.

3. COUNTER-EXAMPLES

So, let's say we have the domain:

$$D = \{ 1, 3, 5, 7, 9, 10 \}$$

the predicate:

$P(x)$ is "x is odd"

and the quantified statement:

$$\forall x \in D, P(x)$$

If we inspect the domain D ,
we can see that not *all* elements are odd – 10 isn't.

Therefore, we can use 10 as the **counter-example** to show
that the quantified statement $\forall x \in D, P(x)$ is **false**.

Notes

You can use a single counter-example to disprove a quantified statement that uses "for all", if you can find an example that makes the predicate false.

3. COUNTER-EXAMPLES

Practice 5: Find a counter-example for the following.

Given the domain $D = \{ 3, 4, 6, 12 \}$,
 $P(x)$ is the predicate, “ x^2 is even”

$$\forall x \in D, P(x)$$

Notes

You can use a single counter-example to disprove a quantified statement that uses “for all”, if you can find an example that makes the predicate false.

3. COUNTER-EXAMPLES

Practice 5: Find a counter-example for the following.

Given the domain $D = \{ 3, 4, 6, 12 \}$,
 $P(x)$ is the predicate, “ x^2 is even”

$$\forall x \in D, P(x)$$

This is false for $P(3)$; $P(3) \rightarrow 3^2 \rightarrow 9$

Notes

You can use a single counter-example to disprove a quantified statement that uses “for all”, if you can find an example that makes the predicate false.

NEGATIONS OF PREDICATES

4. NEGATIONS OF PREDICATES

If we come across a quantified statement that is false, we can say that the *negation* of that statement is true.

$$D = \{ 1, 2, 4, 6, 8, 10 \}$$

E(x) is the predicate, "x is even"

Notes

4. NEGATIONS OF PREDICATES

For example, let's look at this quantified statement again:

$\forall x \in D, E(x)$, with $D = \{1, 2, 4, 6, 8, 10\}$
and $E(x)$ is the predicate, "x is even"

Not every input x in the domain D makes the predicate $E(x)$ result to true, so we **cannot** state that it is "true for all elements of x in D ".

But what do we get if we negate it?

Notes

4. NEGATIONS OF PREDICATES

For example, let's look at this quantified statement again:

$\forall x \in D, E(x)$, with $D = \{1, 2, 4, 6, 8, 10\}$
and $E(x)$ is the predicate, "x is even"

$\neg(\forall x \in D, E(x))$: **It is not true that...**
For all x in D, x is even.

Or,

$\exists x \in D, \neg E(x)$: **There exists some x in D,**
such that x is not even.

Notes

$$\neg(\forall x \in D, P(x)) \\ \equiv \exists x \in D, \neg P(x)$$

4. NEGATIONS OF PREDICATES

Likewise, let's look at a statement that uses "there exists".

$\exists x \in D, O(x)$, with $D = \{2, 4, 6, 8, 10\}$
and $O(x)$ is the predicate, "x is odd"

We can see that in the set, *none of the elements are odd!* – they're all even. So what is the negation of this quantified statement?

Notes

$$\begin{aligned} & \neg(\forall x \in D, P(x)) \\ \equiv & \exists x \in D, \neg P(x) \end{aligned}$$

4. NEGATIONS OF PREDICATES

Likewise, let's look at a statement that uses "there exists".

$\exists x \in D, O(x)$, with $D = \{2, 4, 6, 8, 10\}$
and $O(x)$ is the predicate, "x is odd"

**$\neg(\exists x \in D, O(x))$: It is not true that...
There exists some x in D,
such that x is odd.**

Or,

$\forall x \in D, \neg O(x)$: For all x in D, x is not odd.

Notes

$$\neg(\forall x \in D, P(x)) \\ \equiv \exists x \in D, \neg P(x)$$

$$\neg(\exists x \in D, P(x)) \\ \equiv \forall x \in D, \neg P(x)$$

4. NEGATIONS OF PREDICATES

To summarize...

$$\neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

It is not true that... (for all x in D , $P(x)$)

\equiv There exists some x in D , such that NOT $P(x)$.

$$\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

It is not true that... (there exists some x in D , such that $P(x)$)

\equiv For all x in D , NOT $P(x)$.

Notes

$$\begin{aligned} &\neg(\forall x \in D, P(x)) \\ &\equiv \exists x \in D, \neg P(x) \end{aligned}$$

$$\begin{aligned} &\neg(\exists x \in D, P(x)) \\ &\equiv \forall x \in D, \neg P(x) \end{aligned}$$

4. NEGATIONS OF PREDICATES

Practice 6: Find the negation of the following:

1. $\exists x \in D, P(x)$

2. $\forall x \in D, \neg Q(x)$

Notes

$$\begin{aligned} & \neg(\forall x \in D, P(x)) \\ & \equiv \exists x \in D, \neg P(x) \end{aligned}$$

$$\begin{aligned} & \neg(\exists x \in D, P(x)) \\ & \equiv \forall x \in D, \neg P(x) \end{aligned}$$

4. NEGATIONS OF PREDICATES

Practice 6: Find the negation of the following:

$$\begin{aligned} 1. \quad & \exists x \in D, P(x) \\ & \neg (\exists x \in D, P(x)) \\ & = \forall x \in D, \neg P(x) \end{aligned}$$

$$\begin{aligned} 2. \quad & \forall x \in D, \neg Q(x) \\ & \neg (\forall x \in D, \neg Q(x)) \\ & = \exists x \in D, \neg (\neg Q(x)) \\ & = \exists x \in D, Q(x) \end{aligned}$$

Notes

$$\begin{aligned} & \neg(\forall x \in D, P(x)) \\ & \equiv \exists x \in D, \neg P(x) \end{aligned}$$

$$\begin{aligned} & \neg(\exists x \in D, P(x)) \\ & \equiv \forall x \in D, \neg P(x) \end{aligned}$$

4. NEGATIONS OF PREDICATES

Practice 7: Is this statement true? If not, give the negation in symbolically & in English.

$C(x)$ is the predicate, “ x is a college student”

$T(x)$ is the predicate, “ x is tired all the time”

Domain: The set of all living people.

Quantified Predicate: $\forall x \in D, C(x) \wedge T(x)$

Notes

$$\neg(\forall x \in D, P(x)) \\ \equiv \exists x \in D, \neg P(x)$$

$$\neg(\exists x \in D, P(x)) \\ \equiv \forall x \in D, \neg P(x)$$

4. NEGATIONS OF PREDICATES

Practice 7: Is this statement true? If not, give the negation in symbolically & in English.

$C(x)$ is the predicate, “ x is a college student”

$T(x)$ is the predicate, “ x is tired all the time”

Domain: The set of all living people.

Quantified Predicate: $\forall x \in D, C(x) \wedge T(x)$

Not true for the entire domain; there may be some humans who are not college students, or not tired, or both at the same time...

$$\begin{aligned}\neg(\forall x \in D, C(x) \wedge T(x)) &\equiv \exists x \in D, \neg(C(x) \wedge T(x)) \\ &\equiv \exists x \in D, \neg C(x) \vee \neg T(x)\end{aligned}$$

There exists someone in the set of all humans who is not a college student or not tired.

Notes

$$\begin{aligned}\neg(\forall x \in D, P(x)) \\ \equiv \exists x \in D, \neg P(x)\end{aligned}$$

$$\begin{aligned}\neg(\exists x \in D, P(x)) \\ \equiv \forall x \in D, \neg P(x)\end{aligned}$$

4. NEGATIONS OF PREDICATES

Practice 8: Give the negation in symbolically & in English.

$C(x)$ is the predicate, "x is a college student"

$T(x)$ is the predicate, "x is tired all the time"

Domain: The set of all living people.

Quantified Predicate: $\forall x \in D, C(x) \rightarrow T(x)$

For all people, if x is a college student, then x is tired.

Notes

$$\neg(\forall x \in D, P(x)) \\ \equiv \exists x \in D, \neg P(x)$$

$$\neg(\exists x \in D, P(x)) \\ \equiv \forall x \in D, \neg P(x)$$

4. NEGATIONS OF PREDICATES

Practice 8:

Give the negation in symbolically & in English.

$C(x)$ is the predicate, "x is a college student"

$T(x)$ is the predicate, "x is tired all the time"

Domain: The set of all living people.

Quantified Predicate: $\forall x \in D, C(x) \rightarrow T(x)$

For all people, if x is a college student, then x is tired.

$$\begin{aligned}\neg(\forall x \in D, C(x) \rightarrow T(x)) &\equiv \exists x \in D, \neg(C(x) \rightarrow T(x)) \\ &\equiv \exists x \in D, C(x) \wedge \neg T(x)\end{aligned}$$

There exists some college student, such that x is a college student and x is NOT tired.

Notes

$$\begin{aligned}\neg(\forall x \in D, P(x)) \\ \equiv \exists x \in D, \neg P(x)\end{aligned}$$

$$\begin{aligned}\neg(\exists x \in D, P(x)) \\ \equiv \forall x \in D, \neg P(x)\end{aligned}$$

COMMON NUMERICAL SETS

5. COMMON NUMERICAL SETS

Throughout this class, we will be working with sets, but it is impossible to list all the elements of a set like, say,

“The set of all numbers”.

So, we have some special symbols to denote common numerical sets.

Notes

5. COMMON NUMERICAL SETS

\mathbb{R} The set of all
real numbers

This will be pretty much any number you can think of, including those with infinitely repeating decimals. *Imaginary numbers* (like $\sqrt{-1}$) are not included.

Notes

\mathbb{R} The set of all
real numbers

5. COMMON NUMERICAL SETS

\mathbb{R} The set of all **real numbers**

This will be pretty much any number you can think of, including those with infinitely repeating decimals. *Imaginary numbers* (like $\sqrt{-1}$) are not included.

\mathbb{Q} The set of all **rational numbers**

Any number that can be expressed as a fraction, including whole numbers, as they can be written as $n/1$, as well as fractions that become infinitely repeating decimals like $1/3$.

Notes

\mathbb{R} The set of all **real numbers**

\mathbb{Q} The set of all **rational numbers**

5. COMMON NUMERICAL SETS

\mathbb{R} The set of all
real numbers

This will be pretty much any number you can think of, including those with infinitely repeating decimals. *Imaginary numbers* (like $\sqrt{-1}$) are not included.

\mathbb{Z} The set of all
integers

A number that can be written without a fractional component; whole numbers, including 0 and negative numbers.

\mathbb{Q} The set of all
rational numbers

Any number that can be expressed as a fraction, including whole numbers, as they can be written as $n/1$, as well as fractions that become infinitely repeating decimals like $1/3$.

Notes

\mathbb{R} The set of all
real numbers

\mathbb{Q} The set of all
rational numbers

\mathbb{Z} The set of all
integers

5. COMMON NUMERICAL SETS

\mathbb{R} The set of all **real numbers**

This will be pretty much any number you can think of, including those with infinitely repeating decimals. *Imaginary numbers* (like $\sqrt{-1}$) are not included.

\mathbb{Z} The set of all **integers**

A number that can be written without a fractional component; whole numbers, including 0 and negative numbers.

\mathbb{Q} The set of all **rational numbers**

Any number that can be expressed as a fraction, including whole numbers, as they can be written as $n/1$, as well as fractions that become infinitely repeating decimals like $1/3$.

\mathbb{N} The set of all **natural numbers**

Usually thought of as “counting numbers”, these are whole numbers, including 0, but excluding any negative numbers.

Notes

\mathbb{R} The set of all **real numbers**

\mathbb{Q} The set of all **rational numbers**

\mathbb{Z} The set of all **integers**

\mathbb{N} The set of all **natural numbers**

5. COMMON NUMERICAL SETS

Additionally, we can specify “ ≥ 0 ” or “+” to further restrict a set to “greater than or equal to 0”, or “positive”.

Set	Only positive elements	Only elements 0 or greater
\mathbb{R}	\mathbb{R}^+	$\mathbb{R}^{\geq 0}$
\mathbb{Q}	\mathbb{Q}^+	$\mathbb{Q}^{\geq 0}$
\mathbb{Z}	\mathbb{Z}^+	$\mathbb{Z}^{\geq 0}$
\mathbb{N}	\mathbb{N}^+	$\mathbb{N}^{\geq 0}$

Notes

- \mathbb{R} The set of all **real numbers**
- \mathbb{Q} The set of all **rational numbers**
- \mathbb{Z} The set of all **integers**
- \mathbb{N} The set of all **natural numbers**

NESTED QUANTIFIERS

6. NESTED QUANTIFIERS

Finally, we might also want to write a quantified statement that includes multiple variables. Instead of one input variable, we will have two:

$P(x, y)$ is the predicate, “ $y + 1 = x$ ”.

But when we add another variable, we have to make sure to add another quantifier for it.

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x, y)$$

“For all integers x , there exists some integer y , such that $y + 1 = x$.”

Notes

6. NESTED QUANTIFIERS

“Fred Rogers often told this story about when he was a boy and would see scary things on the news: “My mother would say to me, ‘Look for the helpers. You will always find people who are helping.’”

<https://www.snopes.com/fact-check/look-for-the-helpers/>

Example: Given the set B of all bad events, and the set P of all people, and $H(x, y)$ is the predicate, “ y is helping after x event”.

$\forall x \in B,$

For all bad events x ,

$\exists y \in P,$

There exists some
person y ,

$H(x, y)$

Such that y is helping
after x event.

For all bad events, there exists [at least one] person who helps...

Notes

6. NESTED QUANTIFIERS

Negations: When we have the negation of a quantified statement that includes multiple variables, each part of the statement will be negated:

$$\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x,y))$$

$$\equiv \neg(\forall x \in \mathbb{Z}), \neg(\exists y \in \mathbb{Z}), \neg(P(x,y))$$

$$\equiv \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg P(x,y)$$

Common error: \in stays as \in , and does NOT change into \notin !

Notes

6. NESTED QUANTIFIERS

Practice 7:

Negate the following quantified statement, and specify whether the negation or the original was true.

$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, P(x)$

$P(x)$ is the predicate, " $a + b \in \mathbb{Z}$ "

Noes

6. NESTED QUANTIFIERS

Practice 7:

Negate the following quantified statement, and specify whether the negation or the original was true.

$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, P(x)$

$P(x)$ is the predicate, " $a + b \in \mathbb{Z}$ "

True

"For all integers a and b , $a + b$ is also an integer."

$\neg(\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, P(x))$

$\equiv \exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, \neg P(x)$

False

"There exists an integer a and an integer b , such that $a + b$ is not an integer."

Noes

CONCLUSION

That was a lot of information, so make sure to practice to gain a better grasp of how predicates and quantified statements work!

Next time we will be talking about *implications*, which are essentially “if, then” statements. In discrete math, we can write these either with propositions or with predicates.