# Answer Key

## Grading Scheme

| # | Question | Weight | Points Received |
|---|---|---|---|
| 1 | Cards | 20% | |
| 2 | Cards | 20% | |
| 3 | Dice | 20% | |
| 4 | Expected Value | 20% | |
| 5 | Bernoulli Probability | 20% | |
| 6 | Extra Credit 1 | 2% | |
| 7 | Extra Credit 2 | 2% | |



By Rachel Singh, last updated November 29, 2018

**Cheat sheet**

**Combinatorics - Structures:**

| | Repeats allowed? | Order matters? | Formula |
|---|---|---|---|
| **Permutation**<br>$n$ items to choose from<br>Select $r$ items | no | yes | $P(n,r) = \frac{n!}{(n-r)!}$ |
| **Set**<br>$n$ items to choose from<br>Select $r$ items | no | no | $C(n,r) = \frac{n!}{r!(n-r)!}$ |
| **Ordered list**<br>$n$ items to choose from<br>Select $r$ items | yes | yes | $n^r$ |
| **Unordered list**<br>$n$ different types<br>Select $r$ items | yes | no | $C(n+r-1,n)$ |
| **Unordered list for Binary Strings**<br>$r$ 1's<br>$n-r$ 0's | yes | no | $C(n,r)$ |

**The Sample Set** $S$ is the set of all possible outcomes, without restriction. $n(S)$ is the size of this set.

**The Event** $E$ is the set of outcomes that match some event that we are concerned with, and $n(E)$ is the size of this set.

**The probability of Event** $E$ **occurring** is given by $Prob(E) = \frac{n(E)}{n(S)}$

**Two events are Disjoint** (aka mutually exclusive) if they cannot occur simultaneously. This is handy for the sum rule

**Two events are Independent** if they occurrence of one event is not influenced by the occurrence (or non-occurrence) of the other event. This is handy for the product rule.



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## Cheat sheet

**The General Sum Rule** If $E_1$ and $E_2$ are any events in a given experiment, then the probability that $E_1$ OR $E_2$ occurs is given by:

$$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2)$$

If $E_1$ and $E_2$ are disjoint, then $Prob(E_1 \text{ and } E_2)$ is 0.

**The General Product Rule** If $E_1$ and $E_2$ are any events in a given experiment, then the probability that both $E_1$ AND $E_2$ occur is given by

$$Prob(E_1 \text{ and } E_2) = Prob(E_2) \cdot Prob(E_1|E_2)$$

...which is also equivalent to...

$$Prob(E_1 \text{ and } E_2) = Prob(E_2) \cdot Prob(E_2|E_1)$$

**The probability of $E_1$ given $E_2$** Given events $E_1$ and $E_2$ for some experiment, we define the probability of $E_1$ happening given that $E_2$ occurred as by $Prob(E_1|E_2)$.

**Probability in a Bernoulli Trial** For a Bernoulli trial, we run the trial $n$ times. We're looking for some success criteria to occurr, and we want it to occur exactly $k$ times. The probability of the success occurring is $p$. Given these variables, you can calculate the probability of the success occurrin $k$ times in $n$ trials with:

$$C(n, k) \cdot p^k \cdot (1 - p)^{n-k}$$

**Expected (average) value** For a given experiment with a random number X, where the values that X could be come from the set $\{x_1, x_2, ..., x_n\}$, the expected (average) value of X is:

$$E[X] = x_1 \cdot Prob(X = x_1) + ... + x_n \cdot Prob(X = x_n)$$

**Expected value in a Bernoulli trial** Given a trial performed $n$ times and the probability of success being $p$, the expected value in a Bernoulli trial is:

$$E[X] = n \cdot p$$

**Question 1: Cards part 1** □ 0 □ 1 □ 2 □ 3 □ 4

Given a standard deck of 52 8cards, we're going to draw cards without replacing them back in the deck. Answer the questions for the following scenarios.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A ♣ | 2 ♣ | 3 ♣ | 4 ♣ | 5 ♣ | 6 ♣ | 7 ♣ | 8 ♣ | 9 ♣ | 10 ♣ | J ♣ | Q ♣ | K ♣ |
| A ♠ | 2 ♠ | 3 ♠ | 4 ♠ | 5 ♠ | 6 ♠ | 7 ♠ | 8 ♠ | 9 ♠ | 10 ♠ | J ♠ | Q ♠ | K ♠ |
| A ♢ | 2 ♢ | 3 ♢ | 4 ♢ | 5 ♢ | 6 ♢ | 7 ♢ | 8 ♢ | 9 ♢ | 10 ♢ | J ♢ | Q ♢ | K ♢ |
| A ♡ | 2 ♡ | 3 ♡ | 4 ♡ | 5 ♡ | 6 ♡ | 7 ♡ | 8 ♡ | 9 ♡ | 10 ♡ | J ♡ | Q ♡ | K ♡ |

For the first experiment, you're drawing one card from the deck.

a. The sample size, $n(S)$, is:

b. What is the amount of events, $n(E_1)$, for the result of getting a value of 6:

c. What is the amount of events, $n(E_2)$, for the result of getting a suit of diamonds:

d. What is the amount of events, $n(E_3)$, for the result of getting a 6 of diamonds:

e. Probability of getting a 6, $Prob(E_1)$:

f. Probability of getting a diamond, $Prob(E_2)$:

g. Probability of getting a 6 OR getting a diamond card, $Prob(E_1 \text{ and } E_2)$:

**Question 2: Cards part 2** ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

For the second experiment, you're drawing two cards from a deck.

a. What is the probability of getting an Ace as your first card, $Prob(E_{A1})$?

b. What is the probability of getting a Queen as your second card, given that you got an Ace as your first card? $Prob(E_{Q2}|E_{A1})$?

c. What is the probability of getting an Ace as your second card, given that you got an Ace as your first card? $Prob(E_{A2}|E_{A1})$?

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**Question 3: Dice roll** □ 0 □ 1 □ 2 □ 3 □ 4

In an experiment, we are rolling two dice. All outcomes are:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| 2 | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| 3 | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| 4 | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| 5 | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| 6 | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

Find the probability of the following.

a. Getting an even number on the first die.

b. Getting an even number exactly one die (not both).

c. Getting at least one even number.

d. The second die having a lower value than the first die.

**Question 4: Bernoulli game** □ 0 □ 1 □ 2 □ 3 □ 4

At a carnival game, you can play a coin flip game to earn money. In the game, you flip a coin 3 times. Every time you get a Heads, you get +$5. Fill out the table below. The probability of a Bernoulli trial is $C(n,k) \cdot p^k \cdot (1-p)^{n-k}$.

- Probability of getting $0:
- Probability of getting $5:
- Probability of getting $10:
- Probability of getting $15:

| Outcome | # Heads | Prize value $x_i$ | $Prob(E_i)$ |
|---|---|---|---|
| 1 | 0 | X = $0 | |
| 2 | 1 | X = $5 | |
| 3 | 2 | X = $10 | |
| 4 | 3 | X = $15 | |

Next, calculate the expected average value, $E[X]$, with

$$E[X] = x_1 \cdot Prob(X = x_1) + ... + x_n \cdot Prob(X = x_n)$$

**Question 5: Another Bernoulli trial** □ 0 □ 1 □ 2 □ 3 □ 4

In a die roll experiment, we are rolling a die 10 times. A single success is getting a value of 1 or 2. In this trial, we want to find the probability of getting at least 8 successes. Find the probability.

**Question 6: Extra Credit 1** □ 0 □ 1 □ 2 □ 3 □ 4

A box contains 15 toys. 3 toys in the box are broken. Let's say a kid will take 5 toys from the box...

a. How many ways are there for the kid to select 5 toys with no restrictions?

b. How many ways will there be exactly 1 broken toy?

c. How many ways will contain all the broken toys?

d. How many ways will contain no broken toys?

**Question 7: Extra Credit 2** □ 0 □ 1 □ 2 □ 3 □ 4

In a classroom, there are 10 computer science students, 8 IT students, and 5 math students. A president, vice president, and secretary must be elected. How many ways are there to elect these positions, with the additional constraints:

a. No restrictions - anyone can serve any role.

b. No Computer Science students can be elected.

c. At least one Computer Science student must be elected.

d. Either the president or the vice president must be an IT student, and CS and Math will be chosen for the other roles.

# Answer Key

1. a. $n(S) = 52$
   b. $n(E_1) = 4$
   c. $n(E_2) = 13$
   d. $n(E_3) = 1$
   e. $Prob(E_1) = 4/52$
   f. $Prob(E_2) = 13/52$
   g. $Prob(E_1$ and $E_2) = 4/52 + 13/52$ - $1/52$

   Don't forget the overlap.

2. a. $Prob(E_{A1}) = 4/52$
   b. $Prob(E_{Q2}|E_{A1}) = 4/51$
   c. $Prob(E_{A2}|E_{A1}) = 3/51$

   A lot of people wrote the solution to $Prob(E_{A1}$ and $E_{Q2})$ and $Prob(E_{A1}$ and $E_{A2})$ here instead of the probability of events $E_{Q2}$ or $E_{A2}$ given that $E_{A1}$ occured.

3. a. 18 total outcomes; 18/36
   { (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }
   b. 18 total outcomes; 18/36
   { (1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5) }
   c. 27 total outcomes; 27/36
   { (1,2), (1,4), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,2), (3,4), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,2), (5,4), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }
   c. 15 total outcomes; 15/36
   { (2,1), (3,2), (3,1), (4,3), (4,2), (4,1), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5) }


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4.
- \$0: $C(3,0)(1/2)^0(1/2)^3 = 1 \cdot 1 \cdot \frac{1}{8} = \frac{1}{8}$
- \$5: $C(3,1)(1/2)^1(1/2)^2 = 3 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$
- \$10: $C(3,2)(1/2)^2(1/2)^1 = 3 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$
- \$15: $C(3,3)(1/2)^3(1/2)^0 = 1 \cdot \frac{1}{8} \cdot 1 = \frac{1}{8}$
- E[X]= $0[C(3,0)(1/2)^0(1/2)^3]+5[C(3,1)(1/2)^1(1/2)^2]+10[C(3,2)(1/2)^2(1/2)^1]+15[C(3,3)(1/2)^3(1/2)^0]$
  $= 0(\frac{1}{8}) + 5(\frac{3}{8}) + 10(\frac{3}{8}) + 15(\frac{1}{8}) = \frac{15+30+15}{8} = 7.5$

5.
- 8 successes: $C(10,8)(2/6)^8(4/6)^2$
- 9 successes: $C(10,9)(2/6)^9(4/6)^1$
- 10 successes: $C(10,10)(2/6)^{10}(4/6)^0$
- Result: $C(10,8)(2/6)^8(4/6)^2+C(10,9)(2/6)^9(4/6)^1+C(10,10)(2/6)^{10}(4/6)^0$

  $= 20/6561 + 20/59049 + 1/59049 = 67/19683$

6.
a. $C(15,5)$
b. $C(3,1) \cdot C(12,4)$
c. $C(3,3) \cdot C(12,2)$
d. $C(12,5)$

7.
a. $P(23,3)$
b. $P(13,3)$
c. Rule of Complements: $P(23,3) - P(13,3) = 10626 - 1716 = 8910$
Long way, all results:

| Pres | VP | Sec | Equation |
|---|---|---|---|
| CS | IT/Math | IT/Math | $P(10,1) \cdot P(13,2) = 1560$ |
| IT/Math | CS | IT/Math | $P(10,1) \cdot P(13,2) = 1560$ |
| IT/Math | IT/Math | CS | $P(10,1) \cdot P(13,2) = 1560$ |
| CS | CS | IT/Math | $P(10,2) \cdot P(13,1) = 1170$ |
| CS | IT/Math | CS | $P(10,2) \cdot P(13,1) = 1170$ |
| IT/Math | CS | CS | $P(10,2) \cdot P(13,1) = 1170$ |
| CS | CS | CS | $P(10,3) = 720$ |

Result: $1560 + 1560 + 1560 + 1170 + 1170 + 1170 + 720 = 8910$

d. $P(8,1) \cdot P(15,2) + P(8,1) \cdot P(15,2)$