Probability: Sum and Product Rule

The Sum Rule

Disjoint events

Two events are said to be **disjoint** (or *mutually exclusive* if they cannot occur simultaneously. [a]

[a] From Discrete Math by Ensley and Crawley, page 448

Question 1

For each of the experiments given below, decide if the events described are disjoint:

a. When tossing a coin four times, let E_1 be the event that there are exactly three heads and E_2 be the event that there are exactly two heads.

b. When choosing four cards, let E_1 be the event that the cards have the same value and E_2 be the event that the cards have the same suit.

c. When choosing a committee of three people from a club with 8 men and 12 women, let E_1 be the event that the committee has a woman and let E_2 be the event that the committee has a man.

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The Sum Rule, Theorem 1 [a]

If E_1 and E_2 are disjoint events in a given experiment, then the probability that E_1 or E_2 occurs is the sum of $Prob(E_1)$ and $Prob(E_2)$. That is,

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2)$$

for disjoint events.

[a]From Discrete Math by Ensley and Crawley, page 449

Question 2

A card is drawn from an ordinary deck of 52 cards. Show how to use the basic sum rule to find the probability that the card is an ace or a jack.

- When picking one card out of a deck, the amount of total outcomes, $n(S)$, is 52.
- Event 1 is getting an ace.
  $E_1 = \{A\heartsuit, A\diamondsuit, A\spadesuit, A\clubsuit\}$

  $n(E_1) =$

  $Prob(E_1) = \frac{n(E_1)}{n(S)} =$
- Event 2 is getting a jack.
  $E_2 = \{J\heartsuit, J\diamondsuit, J\spadesuit, J\clubsuit\}$

  $n(E_2) =$

  $Prob(E_2) = \frac{n(E_2)}{n(S)} =$
- Probability of Event 1 occuring, or Event 2 occurring...

  $Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) =$

The General Sum Rule, Theorem 2 [a]

If E_1 and E_2 are any events in a given experiment, then the probability that E_1 or E_2 occurs is given by

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 and E_2)$$

If E_1 and E_2 are disjoint, then $E_1 \cap E_2 = \emptyset$, so $Prob(E_1 and E_2) = 0$.

[a] From Discrete Math by Ensley and Crawley, page 450

Question 3

A card is drawn from an ordinary deck of 52 cards. Show how to use the general sum rule to find the probability that the card is a diamond or a red face card (jack, queen, king).

- Event 1 is getting a diamond.
  $E_1 = \{\diamondsuit A, \diamondsuit 2, \diamondsuit 3, \diamondsuit 4, \diamondsuit 5, \diamondsuit 6, \diamondsuit 7, \diamondsuit 8, \diamondsuit 9, \diamondsuit 10, \diamondsuit J, \diamondsuit Q, \diamondsuit K\}$

  $n(E_1) =$

  $Prob(E_1) =$

- Event 2 is getting a face card.
  $E_2 = \{\diamondsuit J, \diamondsuit Q, \diamondsuit K, \heartsuit J, \heartsuit Q, \heartsuit K, \spadesuit J, \spadesuit Q, \spadesuit K, \clubsuit J, \clubsuit Q, \clubsuit K\}$

  $n(E_2) =$

  $Prob(E_2) =$

- Getting Event 1 and Event 2 at the same time...
  E_1 and $E_2 = \{\diamondsuit J, \diamondsuit Q, \diamondsuit K\}$

  $n(E_1 \text{ and } E_2) =$

  $Prob(E_1 \text{ and } E_2) =$

- Probability of getting Event 1 OR Event 2...
  $Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2) =$

Question 4
A card is drawn from an ordinary deck of 52 cards. Show how to use the general sum rule to find the probability that the card is an even number value or a black face card (jack, queen, king).

- Event 1 is

  $n(E_1) =$

  $Prob(E_1) =$

- Event 2 is

  $n(E_2) =$

  $Prob(E_2) =$

- Getting Event 1 and Event 2 at the same time...

  $n(E_1 \text{ and } E_2) =$

  $Prob(E_1 \text{ and } E_2) =$

- Probability of getting Event 1 OR Event 2...
  $Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2) =$

Question 5
What is the probability that when a pair of dice are rolled, either (at least) one die shows a 5 or the dice sum to 8?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

a. What is the sample size? $n(S) =$

b. How many outcomes $n(E_1)$ are there where you get either 5 for the first die, 5 for the second die, or 5 for both dice? Write out the set of E_1.

c. How many outcomes $n(E_2)$ are there where the two dice sum to 8? Write out the set of E_2.

d. What is the amount of overlap $n(E_1 \quad AND \quad E_2)$? Write out this set.

e. Use the General Sum Rule to find the probability that you will get either at least one die showing a 5, OR the dice sum to 8.

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This given that

The probability of E_1 given E_2 [a]

Given events E_1 and E_2 for some experiment, we define the probability of E_1 given E_2, denoted by $Prob(E_1|E_2)$, as the probability that E_1 happens given that E_2 occurs. Note that if E_1 and E_2 are independent, then $Prob(E_1|E_2) = Prob(E_1)$.

[a] From Discrete Math by Ensley and Crawley, page 452

Question 6

You're drawing two cards from a deck, without replacement. E_1 occurs first, and E_2 occurs second. Find the probability of $Prob(E_2|E_1)$ (Probability of the 2nd event occuring, given that the 1st even has occured).

E_1 is that you draw a face card.

E_2 is that you draw an ace.

Hint

You don't need to calculate $Prob(E_1)$; the main idea is how does E_1 occuring affect the state of the sample space? The answer here is that there is one less card in the deck, so $Prob(E_2|E_1)$ is the probability of getting an ace in a deck with 51 cards in it.

The Product Rule

Independent events [a]

Two events are said to be **independent** if the occurrence of one event is not influenced by the occurred (or nonoccurrence) of the other event.

[a] From Discrete Math by Ensley and Crawley, page 451

Question 7

For each of the following experiments given below, decide if the events described are independent:

a. When rolling a 6-sided die four times, let E_1 be the event that the first two rolls sum to 7 and let E_2 be the event that the last two rolls sum to 10.
Are these events independent or dependent?

b. At a party, there are 15 guests - 6 boys, 7 girls, and 2 nonbinary children. There is a prize drawing for 2 prizes. Once a person wins a prize, they cannot win a second prize. E_1 is the event that a boy wins a prize, and E_2 is the event that a girl wins a prize.
Are these events independent or dependent?

c. At the same party as in (b.), one child can win multiple prizes (they are not removed from the selection pool if they win a prize.) E_1 is the event that a boy wins a prize, and E_2 is the event that a girl wins a prize.
Are these events independent or dependent?

The Product Rule, Theorem 3 [a]

If E_1 and E_2 are independent events in a given experiment, then the probability that both E_1 and E_2 occur is the product of $Prob(E_1$ and $Prob(E_2$. That is,

$$Prob(E_1 and E_2) = Prob(E_1) \cdot Prob(E_2)$$

for independent events.

[a] From Discrete Math by Ensley and Crawley, page 452

Question 8

Suppose I have a "loaded" die for which the probability of a 6 appearing is $\frac{1}{2}$, while the probability of each of the other faces appearing is $\frac{1}{10}$. What is the probability of getting a 5 and then a 6 on two tosses of the loaded die?

First identify E_1 and E_2. These events are independent, so you can use the Product Rule to find $Prob(E_1 and E_2)$.

The General Product Rule, Theorem 4 [a]

If E_1 and E_2 are any events in a given experiment, then the probability that both E_1 and E_2 occur is given by

$$\begin{aligned} Prob(E_1 and E_2) &= Prob(E_2) \cdot Prob(E_1 | E_2) \\ &= Prob(E_1) \cdot Prob(E_2 | E_1) \end{aligned}$$

Note that if E_1 and E_2 are independent, then this says the same thing as Theorem 3.

[a] From Discrete Math by Ensley and Crawley, page 453

Question 9

Two marbles are chosen from a bag containing three red, five white, and eight green marbles, so there are 16 total marbles. What is the probability that both are red?

Here, the event R_1 is "the first marble is red", and the event R_2 is "the second marble is red".

What is $Prob(R_1)$?

Since R_2 depends on R_1 occurring, after R_1 occurs, there are 15 marbles left. One red marble has been selected, so there are 2 red marbles left.

What is $Prob(R_2 | R_1)$?

With this information, what is $Prob(R_1 and R_2)$?

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Question 10
Two marbles are chosen from a bag containing three red, five white, and eight green marbles, so there are 16 total marbles. What is the probability that one is white and one is green?

Let's say we have the events W_1 (White first), W_2 (White second), G_1 (Green first), and G_2 (Green second), so we can get our result in two ways: with (W_1, G_2) **OR** with(G_1, W_2), so you can calculate the result as

$$Prob(W_1 and G_2) + Prob(G_1 and W_2)$$