Extra: Game States and Matrices

The Gambler's Ruin Problem

The Gambler's Ruin Problem
Page 478 of the book highlights a game played between two characters: Player "H" and Player "T". Each character begins with a certain amount of markers (or tokens), and they play by flipping a **coin**. Whenever one of them loses a "round", they give one marker to their opponent.

If a *heads* is flipped, then Player H wins a marker from Player T. For a *tails*, Player T wins a marker from Player H. The game is over once somebody is out of markers.

Game setup: For the game we'll be talking about in thie example, the rules are:

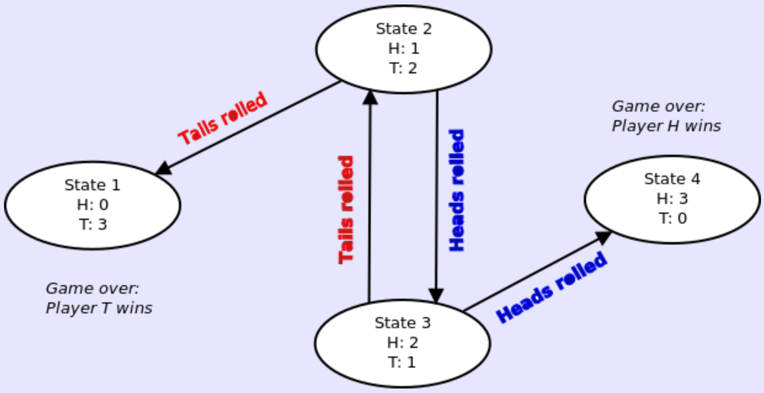
- There are 3 total markers (so one player will have more markers than the other)
- Once somebody is out of markers, the game is over.
- Each coin flip, the loser gives one of their markers to the other player. (One gains, one loses)

Game states: Before we start modeling the game with a matrix, let's map out all the game states. We won't worry about what is the beginning state, and each coin flip, one person gains a marker and one person loses a marker. There are four possible states in the game:

State #	H's markers	T's markers
1	0	3
2	1	2
3	2	1
4	3	0

By Rachel Singh, last updated November 13, 2018

State diagram: Given these states, and the fact that each coin flip one person loses a marker and gives it to the other, our state diagram would look like:



State change matrix: Now we will build out a matrix to show the probability of switching between states.

The matrix will be 4×4. Each **row** will be a state, and each cell in that row is the probability of going from that state to a new state.

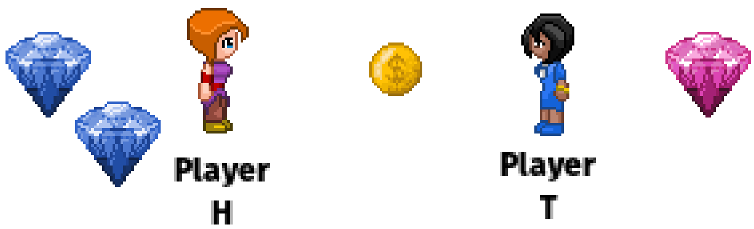
	State 1	State 2	State 3	State 4
State 1 $\rightarrow$	1	0	0	0
State 2 $\rightarrow$	1/2	0	1/2	0
State 3 $\rightarrow$	0	1/2	0	1/2
State 4 $\rightarrow$	0	0	0	1

This matrix is in the format, "from *row* state to *col* state". In Row 2, the probability of going from (State 2 $\rightarrow$ State 1) is 1/2, because the coin flip has a half chance of being heads, and a half chance of being tails.

State 1 and State 4 they are gameover states: you cannot move from State 1 to another state, so it has a 1 in that cell.

Question 1

Let's say a game starts where Player H has 2 markers and player 1 has 1 marker. This is state 3.



What is the probability of...

a. Going from State 3 to State 1?

b. Going from State 3 to State 2?

c. Going from State 3 to State 3?

d. Going from State 3 to State 4?

Transition Matrix

If you have a game with states 1 through n, then your transition matrix M is

$$M_{i,j} = Prob(\text{the game changes from state } i \text{ to state } j \text{ in one move}).$$

[a]

[a] Discrete Structures, Ensley and Crawley

Question 2

Let's say in the game there are 2 total markers instead of three.

a. What are all the states in the game?

b. Draw the transition matrix for this game.

Transition Matrix
Given the matrices M and N, where the amount of *rows* in M is the same amount of *columns* in N, we can find the product $P = M \cdot N$ where each entry at row i, column j of P is the row-column product of row i from M and column j from N. In other words, [a]

$$P_{i,j} = M_{i,1} \cdot N_{1,j} + M_{i,2} \cdot N_{2,j} + ...$$

[a] Discrete Structures, Ensley and Crawley

Question 3
Calculate the product $M \cdot M$ (aka M^2) for our original game with 3 markers.

	Col 1	Col 2	Col 3	Col 4
Row 1	1	0	0	0
Row 2	1/2	0	1/2	0
Row 3	0	1/2	0	1/2
Row 4	0	0	0	1

The result ends up being the probability that the game processes from state i to state j in **two** moves.

Work on the next page...

Formula: $P_{i,j} = M_{i,1} \cdot N_{1,j} + M_{i,2} \cdot N_{2,j} + M_{i,3} \cdot N_{3,j} + M_{i,4} \cdot N_{4,j}$
i is the row and j is the column.

First row:

	$M_{i,1} \cdot N_{1,j}$	$M_{i,2} \cdot N_{2,j}$	$M_{i,3} \cdot N_{3,j}$	$M_{i,4} \cdot N_{4,j}$	Result
$M^2_{1,1} =$ $(i = 1, j = 1)$	$M_{1,1} \cdot M_{1,1}$	$M_{1,2} \cdot M_{2,1}$	$M_{1,3} \cdot M_{3,1}$	$M_{1,4} \cdot M_{4,1}$	
$M^2_{1,2} =$ $(i = 1, j = 2)$	$M_{1,1} \cdot M_{1,2}$	$M_{1,2} \cdot M_{2,2}$	$M_{1,3} \cdot M_{3,2}$	$M_{1,4} \cdot M_{4,2}$	
$M^2_{1,3} =$ $(i = 1, j = 3)$	$M_{1,1} \cdot M_{1,3}$	$M_{1,2} \cdot M_{2,3}$	$M_{1,3} \cdot M_{3,3}$	$M_{1,4} \cdot M_{4,3}$	
$M^2_{1,4} =$ $(i = 1, j = 4)$	$M_{1,1} \cdot M_{1,4}$	$M_{1,2} \cdot M_{2,4}$	$M_{1,3} \cdot M_{3,4}$	$M_{1,4} \cdot M_{4,4}$	

Second row:

	$M_{i,1} \cdot N_{1,j}$	$M_{i,2} \cdot N_{2,j}$	$M_{i,3} \cdot N_{3,j}$	$M_{i,4} \cdot N_{4,j}$	Result
$M^2_{2,1} =$ $(i = 2, j = 1)$	$M_{2,1} \cdot M_{1,1}$	$M_{2,2} \cdot M_{2,1}$	$M_{2,3} \cdot M_{3,1}$	$M_{2,4} \cdot M_{4,1}$	
$M^2_{2,2} =$ $(i = 2, j = 2)$	$M_{2,1} \cdot M_{1,2}$	$M_{2,2} \cdot M_{2,2}$	$M_{2,3} \cdot M_{3,2}$	$M_{2,4} \cdot M_{4,2}$	
$M^2_{2,3} =$ $(i = 2, j = 3)$	$M_{2,1} \cdot M_{1,3}$	$M_{2,2} \cdot M_{2,3}$	$M_{2,3} \cdot M_{3,3}$	$M_{2,4} \cdot M_{4,3}$	
$M^2_{2,4} =$ $(i = 2, j = 4)$	$M_{2,1} \cdot M_{1,4}$	$M_{2,2} \cdot M_{2,4}$	$M_{2,3} \cdot M_{3,4}$	$M_{2,4} \cdot M_{4,4}$	

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Formula: $P_{i,j} = M_{i,1} \cdot N_{1,j} + M_{i,2} \cdot N_{2,j} + M_{i,3} \cdot N_{3,j} + M_{i,4} \cdot N_{4,j}$
i is the row and j is the column.

Third row:

	$M_{i,1} \cdot N_{1,j}$	$M_{i,2} \cdot N_{2,j}$	$M_{i,3} \cdot N_{3,j}$	$M_{i,4} \cdot N_{4,j}$	Result
$M^2_{3,1} =$ $(i = 3, j = 1)$	$M_{3,1} \cdot M_{1,1}$	$M_{3,2} \cdot M_{2,1}$	$M_{3,3} \cdot M_{3,1}$	$M_{3,4} \cdot M_{4,1}$	
$M^2_{3,2} =$ $(i = 3, j = 2)$	$M_{3,1} \cdot M_{1,2}$	$M_{3,2} \cdot M_{2,2}$	$M_{3,3} \cdot M_{3,2}$	$M_{3,4} \cdot M_{4,2}$	
$M^2_{3,3} =$ $(i = 3, j = 3)$	$M_{3,1} \cdot M_{1,3}$	$M_{3,2} \cdot M_{2,3}$	$M_{3,3} \cdot M_{3,3}$	$M_{3,4} \cdot M_{4,3}$	
$M^2_{3,4} =$ $(i = 3, j = 4)$	$M_{3,1} \cdot M_{1,4}$	$M_{3,2} \cdot M_{2,4}$	$M_{3,3} \cdot M_{3,4}$	$M_{3,4} \cdot M_{4,4}$	

Fourth row:

	$M_{i,1} \cdot N_{1,j}$	$M_{i,2} \cdot N_{2,j}$	$M_{i,3} \cdot N_{3,j}$	$M_{i,4} \cdot N_{4,j}$	Result
$M^2_{4,1} =$ $(i = 4, j = 1)$	$M_{4,1} \cdot M_{1,1}$	$M_{4,2} \cdot M_{2,1}$	$M_{4,3} \cdot M_{3,1}$	$M_{4,4} \cdot M_{4,1}$	
$M^2_{4,2} =$ $(i = 4, j = 2)$	$M_{4,1} \cdot M_{1,2}$	$M_{4,2} \cdot M_{2,2}$	$M_{4,3} \cdot M_{3,2}$	$M_{4,4} \cdot M_{4,2}$	
$M^2_{4,3} =$ $(i = 4, j = 3)$	$M_{4,1} \cdot M_{1,3}$	$M_{4,2} \cdot M_{2,3}$	$M_{4,3} \cdot M_{3,3}$	$M_{4,4} \cdot M_{4,3}$	
$M^2_{4,4} =$ $(i = 4, j = 4)$	$M_{4,1} \cdot M_{1,4}$	$M_{4,2} \cdot M_{2,4}$	$M_{4,3} \cdot M_{3,4}$	$M_{4,4} \cdot M_{4,4}$	

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$M^2 =$

	Col 1	Col 2	Col 3	Col 4
Row 1	$M_{1,1}$	$M_{1,2}$	$M_{1,3}$	$M_{1,4}$
Row 2	$M_{2,1}$	$M_{2,2}$	$M_{2,3}$	$M_{2,4}$
Row 3	$M_{3,1}$	$M_{3,2}$	$M_{3,3}$	$M_{3,4}$
Row 4	$M_{4,1}$	$M_{4,2}$	$M_{4,3}$	$M_{4,4}$

Fill out the matrix:

$M^2 =$

	Col 1	Col 2	Col 3	Col 4
Row 1				
Row 2				
Row 3				
Row 4				

You should get this:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$