Sum and Product Rules for Probability

Written by Rachel J. Morris, last updated Feb 20, 2018

About

We can use the Rule of Sums and the Rule of Products with probability, but we also need to take into account whether events are Disjoint and whether they are Independent…

Topics

1. Disjoint Events
2. The Sum Rule

   and the General Sum Rule
3. Independent Events
4. The probability of

   this given that
5. The Product Rule

   and the General Product Rule
6. Bernoulli Trials

Disjoint Events

1. Disjoint Events

Two events are said to be *disjoint* (or *mutually exclusive*) if they cannot occur simultaneously.

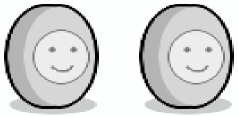
From Discrete Mathematics, Ensley & Crawley, page 448

This will come into play when we have two events, and we want to calculate the probability of <u>this</u> "or" <u>that</u> occurring. We will have overlap, if they are *not disjoint*.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

1. Disjoint Events

Experiment: Flipping two coins 

Event 1, E_1: Get two heads

Event 2, E_2: Get two tails

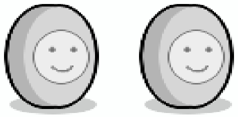
Are these two events **Disjoint?**

Notes

Two events are **disjoint** if they cannot occur simultaneously.

1. Disjoint Events

Experiment: Flipping two coins



Event 1, E_1: Get two heads

Event 2, E_2: Get two tails

Are these two events **Disjoint?**

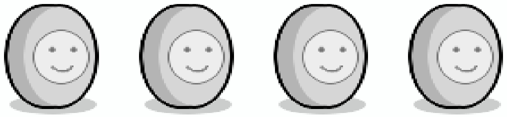
Yes, they are disjoint because we cannot get both two heads and two tails when just flipping two coins.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

1. Disjoint Events

Experiment: Flipping four coins



Event 1, E_1: Get two heads

Event 2, E_2: Get two tails

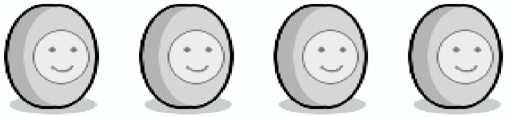
Are these two events **Disjoint?**

Notes

Two events are **disjoint** if they cannot occur simultaneously.

1. Disjoint Events

Experiment: Flipping four coins



Event 1, E_1: Get two heads

Event 2, E_2: Get two tails

Are these two events **Disjoint?**

No, they are not disjoint because we could get two heads AND two tails when we flip 4 or more coins.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

1. Disjoint Events

Experiment: Drawing a card from a deck

Event 1, E_1: Get an ace

Event 2, E_2: Getting a diamond

Are these two events **Disjoint?**

Notes

Two events are **disjoint** if they cannot occur simultaneously.

1. Disjoint Events

Experiment: Drawing a card from a deck

Event 1, E_1: Get an ace

Event 2, E_2: Getting a diamond

Are these two events **Disjoint?**

No, we can get an ace of diamonds.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
& the General Sum Rule

2. The Sum Rules

Theorem 1 – The Sum Rule
If E_1 and E_2 are **disjoint** events in a given experiment, then the probability that E_1 or E_2 occurs is the sum of $Prob(E_1)$ and $Prob(E_2)$.

From Discrete Mathematics, Ensley & Crawley, page 449

We will use the General Sum Rule (covered in a bit) for probabilities where the outcomes are *not* disjoint.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Basic Sum Rule
$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2)$

2. The Sum Rules

We are drawing a card from a standard deck of 52 cards. What is the probability that we get **a Jack or a Queen?**

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Basic Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2)

2. The Sum Rules

We are drawing a card from a standard deck of 52 cards. What is the probability that we get **a Jack or a Queen?**

$n(S) = 52$

E_1 = Getting a Jack = { ♠J, ♥J, ♦J, ♣J }

$n(E_1) = 4$ $\quad$ $Prob(E_1) = 4/52$

E_2 = Getting a Queen = { ♠Q, ♥Q, ♦Q, ♣Q }

$n(E_2) = 4$ $\quad$ $Prob(E_2) = 4/52$

$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) = 8/52 \quad = 2/13$

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Basic Sum Rule

$Prob(E_1 \text{ or } E_2) =$

$Prob(E_1) + Prob(E_2)$

2. The Sum Rules

Theorem 2 – The General Sum Rule
If E_1 and E_2 are any events in a given experiment, then the probability that E_1 or E_2 occurs is given by

$$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$$

From Discrete Mathematics, Ensley & Crawley, page 450

We can use this in the case where our events are not disjoint. If we use it on disjoint sets, it will turn into the basic sum rule.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

2. The Sum Rules

We are drawing a card from a standard deck. What is the probability of getting **a Queen or a Diamond?**

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) =$
$\text{Prob}(E_1) + \text{Prob}(E_2)$
$- \text{Prob}(E_1 \text{ and } E_2)$

2. The Sum Rules

We are drawing a card from a standard deck. What is the probability of getting **a Queen or a Diamond?**

E_1 = getting a queen, $n(E_1) = 4$; Prob(E_1) = 4/52

E_2 = getting a diamond, $n(E_2) = 13$; Prob(E_2) = 13/52

Overlap = getting a queen of diamonds = 1/52

Prob(E_1 or E_2) = 4/52 + 13/52 – 1/52 = 16/52 = 4/13

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) – Prob(E_1 and E_2)

Independent Events

3. Independent Events

Two events are said to be *independent* if the occurrence of one event is not influenced by the occurrence (or nonoccurrence) of the other event.

From Discrete Mathematics, Ensley & Crawley, page 451

Think about drawing two cards from a deck. For the first card, we have 52 cards to choose from. For the second card, we would only have 51 cards to choose from.

On the flip side, if you roll a die twice, the first roll does not affect the second role.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

3. Independent Events

Which of the following experiments have independent events?

1) Flipping a coin twice
Event 1: Get one “H” Event 2: Get one “T”

2) Rolling a red and green die.
Event 1: Red is 5 Event 2: Sum is 8

3) Rolling a red and green die.
Event 1: Red is 2 Event 2: Green is 3

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

3. Independent Events

Which of the following experiments have independent events?

1) Flipping a coin twice
Event 1: Get one "H" Event 2: Get one "T"
Independent

2) Rolling a red and green die.
Event 1: Red is 5 Event 2: Sum is 8
Not Independent

3) Rolling a red and green die.
Event 1: Red is 2 Event 2: Green is 3
Independent

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) =$
$\text{Prob}(E_1) + \text{Prob}(E_2)$
$- \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Probability of This given That

4. The Probability of This given That

Given events E_1 and E_2 for some experiment, we define the probability of E_1 given E_2, denoted by $Prob(E_1|E_2)$, as the probability that if E_1 happens given that E_2 occurs.

From Discrete Mathematics, Ensley & Crawley, page 452

Again, relating to events that are not independent – if you are picking two cards from a deck, choosing that first card affects the second pic because you've reduced the deck size.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

4. The Probability of This given That

When selecting two cards from a deck, the first card you select will be taken during event 1, and the second card during event 2.

E_1 is getting an Ace E_2 is getting a Jack

What is the probability $Prob(E_2|E_1)$? (The probability of E_2 occurring, given that E_1 has occurred?)

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

4. The Probability of This given That

When selecting two cards from a deck, the first card you select will be taken during event 1, and the second card during event 2.

E_1 is getting an Ace E_2 is getting a Jack

What is the probability Prob(E_2|E_1)? (The probability of E_2 occurring, given that E_1 has occurred?)

After E_1 is chosen, the deck size is 51. So,

Prob(E_2|E_1) = 4/51

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) – Prob(E_1 and E_2)

Two events are **independent** if one does not influence the other.

The Product Rule
& the General Product Rule

5. The Product Rule

If E_1 and E_2 are independent events in a given experiment, then the probability that both E_1 and E_2 occur is the product of $Prob(E_1)$ and $Prob(E_2)$.

$$Prob(E_1 \text{ and } E_2) = Prob(E_1) * Prob(E_2)$$

From Discrete Mathematics, Ensley & Crawley, page 452

If the occurrence of one event doesn't affect the probability for the other event, then you can use this rule. Otherwise, you need the general product rule.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Basic Product Rule
$Prob(E_1 \text{ and } E_2) = Prob(E_1) * Prob(E_2)$

5. The Product Rule

You are rolling a die twice. What is the probability that **you get one 5 and one 6?**

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Basic Product Rule
$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_1) * \text{Prob}(E_2)$

5. The Product Rule

You are rolling a die twice. What is the probability that **you get one 5 and one 6?**

- **S = 36**
- **E_1 = get one 5, n(E_1) = 10** **Prob(E_1) = 10/36 = 5/18**
- **E_2 = get one 6, n(E_2) = 10** **Prob(E_2) = 10/36 = 5/18**

Die A Die B	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) – Prob(E_1 and E_2)

Two events are **independent** if one does not influence the other.

The Basic Product Rule
Prob(E_1 and E_2) = Prob(E_1) * Prob(E_2)

5. The Product Rule

You are rolling a die twice. What is the probability that **you get one 5 and one 6?**

- **S = 36**
- **E_1 = get one 5, $n(E_1)$ = 10** **Prob(E_1) = 10/36 = 5/18**
- **E_2 = get one 6, $n(E_2)$ = 10** **Prob(E_2) = 10/36 = 5/18**

Prob(E_1 and E_2) = 5/18 * 5/18 = 25/324

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) – Prob(E_1 and E_2)

Two events are **independent** if one does not influence the other.

The Basic Product Rule
Prob(E_1 and E_2) = Prob(E_1) * Prob(E_2)

5. The Product Rule

If E_1 and E_2 are any events in a given experiment, then the probability that both E_1 and E_2 occur is given by

$$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_1) * \text{Prob}(E_2|E_1)$$

or

$$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_2) * \text{Prob}(E_1|E_2)$$

From Discrete Mathematics, Ensley & Crawley, page 453

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Product Rule
$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_1) * \text{Prob}(E_2|E_1) = \text{Prob}(E_2) * \text{Prob}(E_1|E_2)$

5. The Product Rule

You're choosing two marbles out of a bag. The bag has 3 red, 5 white, and 8 green marbles. What is the probability that both are red?

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$Prob(E_1 \text{ or } E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Product Rule
$Prob(E_1 \text{ and } E_2) = Prob(E_1) * Prob(E_2|E_1) = Prob(E_2) * Prob(E_1|E_2)$

5. The Product Rule

You're choosing two marbles out of a bag. The bag has 3 red, 5 white, and 8 green marbles. What is the probability that both are red?

E_1 = red marble for first one **Prob(E_1) = 3/16**

E_2 = red marble for second one **Prob($E_2|E_1$) = 2/15**

When you remove one marble from the bag, it affects the sample space for when you select the second.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) – Prob(E_1 and E_2)

Two events are **independent** if one does not influence the other.

The Product Rule
Prob(E_1 and E_2) = Prob(E_1) * Prob($E_2|E_1$) = Prob(E_2) * Prob($E_1|E_2$)

5. The Product Rule

You're choosing two marbles out of a bag. The bag has 3 red, 5 white, and 8 green marbles. What is the probability that both are red?

E_1 = red marble for first one Prob(E_1) = 3/16

E_2 = red marble for second one Prob($E_2|E_1$) = 2/15

So the outcome of both will be…

Prob(E_1 and E_2) = Prob(E_1) * Prob($E_2|E_1$)

= 3/16 * 2/15 = 1/40

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) – Prob(E_1 and E_2)

Two events are **independent** if one does not influence the other.

The Product Rule
Prob(E_1 and E_2) = Prob(E_1) * Prob($E_2|E_1$) = Prob(E_2) * Prob($E_1|E_2$)

Bernoulli Trials

6. Bernoulli Trials

Bernoulli trials are common types of ‘games’, and we will see them used in 6.3 and 6.4.

In the theory of probability and statistics, a Bernoulli trial (or binomial trial) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

From Wikipedia, https://en.wikipedia.org/wiki/Bernoulli_trial

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Product Rule
$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_1) * \text{Prob}(E_2|E_1) = \text{Prob}(E_2) * \text{Prob}(E_1|E_2)$

6. Bernoulli Trials

An example of a Bernoulli trial is just tossing a simple coin. You can classify one of the outcomes as a “Success”, and the other one is a “Fail”.

Heads = Success **Tails = Fail**

Or, you can roll a die and say that a desired outcome, “get a 6”, is a success, and all others are failures

Get a 6 = Success **Get 1 – 5 = Fail**

We will look at these more in-depth next time.

Notes

Two events are **disjoint** if they cannot occur simultaneously.

The Sum Rule
$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)$

Two events are **independent** if one does not influence the other.

The Product Rule
$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_1) * \text{Prob}(E_2|E_1) = \text{Prob}(E_2) * \text{Prob}(E_1|E_2)$

Conclusion

This section has a lot of information, and it builds off of Chapter 5. Make sure you get enough practice with the topics here!