Expected Value in Games of Chance

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About

When we have some probability associated with several outcomes, we can estimate the expected outcome we will receive. This is essentially just the average value given all the outcomes and their weights.

Topics

1. Average Value
2. Expectation in Bernoulli Trials

Average Value

1. Average Value

For a given probability experiment, let *X* be a random variable whose possible values come from the set of numbers { x_1, …, x_n }. Then the <u>expected value</u> of *X*, denoted by *E[X]*, is the sum

$$x_1 * \text{Prob}(X = x_1) + x_2 * \text{Prob}(X = x_2) + \ldots + x_n * \text{Prob}(X = x_n)$$

This is sometimes called the <u>average value</u> of the random variable, thinking of the average of the values *X* takes on over many repetitions of the experiment.

From Discrete Mathematics, Ensley & Crawley, page 467

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

1. Average Value

As an example, let's roll a fair die. We can get 1 through 6, and we have 1/6 chance of getting any value.

Let's write a program to run this trial as a simulation, and average together the values received from the virtual die rolls.

Then, we will calculate the expected value using the formula and compare.

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + … + xn * Prob(X = xn)

1. Average Value

Python Code

```
import random

def RollDie():
    return random.randint( 1, 6 )

# ---------------------------------------------------- #

rolls = int( input( "Roll the die how many times? " ) )

sum = 0
count = 1
for i in range( rolls ):
    roll = RollDie()
    sum += roll

    print( "Roll " + str( count ) + ": " + str( roll ) )
    count += 1

average = float( sum ) / float( rolls )

print( "" )
print( "Rolled " + str( rolls ) + " times." )
print( "Average value from rolls: " + str( average ) )
```

In this code, we have a function that “rolls” the dice, and we run the trial any amount of times.

Once we’re done with the trials, we average the values together.

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

1. Average Value

```
Terminal
File  Edit  View  Search  Terminal  Help
Roll the die how many times? 5
Roll 1: 3
Roll 2: 6
Roll 3: 6
Roll 4: 4
Roll 5: 1

Rolled 5 times.
Average value from rolls: 4.0
```

```
Terminal
File  Edit  View  Search  Terminal  Help
Roll the die how many times? 5
Roll 1: 6
Roll 2: 6
Roll 3: 5
Roll 4: 3
Roll 5: 3

Rolled 5 times.
Average value from rolls: 4.6
```

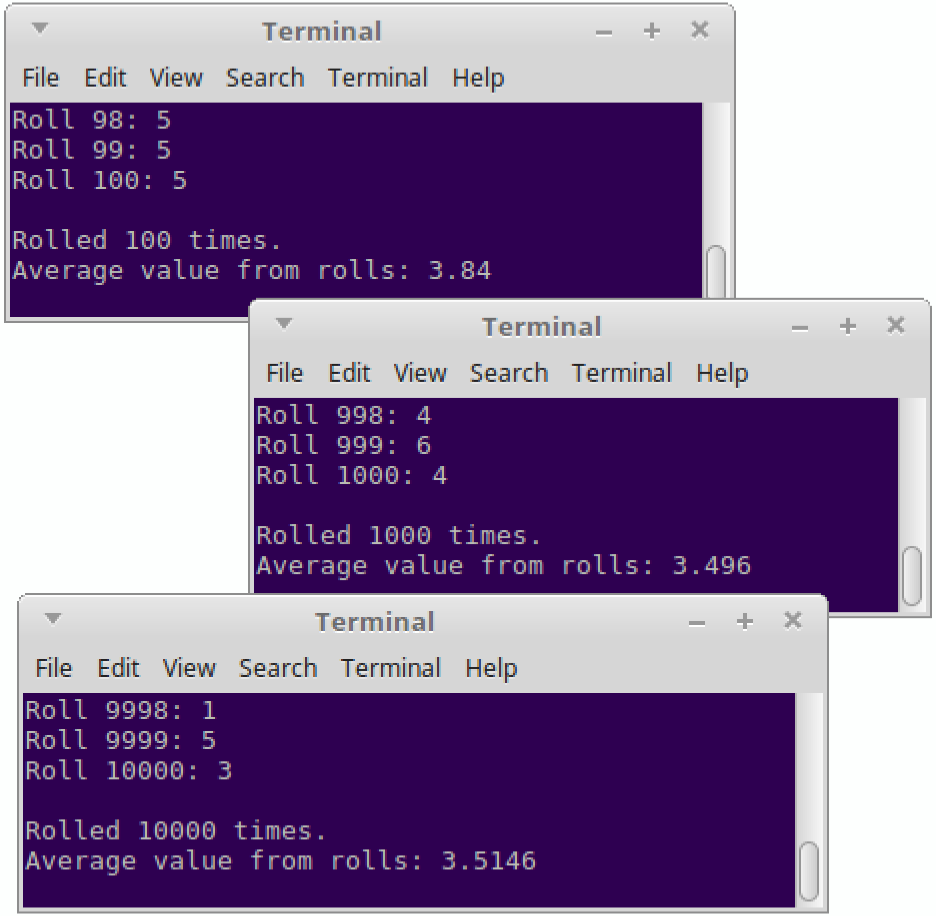
We can run the program multiple times and we will get different average values each time because of the randomness.

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

1. Average Value



But as we roll the die more frequently, we converge on some value…

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + … + xn * Prob(X = xn)

1. Average Value

Now let's calculate the average value.

$$E[X] = x_1 \cdot Prob(X = x_1) + x_2 \cdot Prob(X = x_2) + x_3 \cdot Prob(X = x_3) + x_4 \cdot Prob(X = x_4) + x_5 \cdot Prob(X = x_5) + x_6 \cdot Prob(X = x_6)$$

Outcome	Value x_n	$Prob(X = x_n)$
x_1	1	1/6
x_2	2	1/6
x_3	3	1/6
x_4	4	1/6
x_5	5	1/6
x_6	6	1/6

Notes

Expected Value
E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

1. Average Value

Now let's calculate the average value.

$$E[X]=x_1\cdot Prob(X=x_1)+x_2\cdot Prob(X=x_2)+x_3\cdot Prob(X=x_3)+x_4\cdot Prob(X=x_4)+x_5\cdot Prob(X=x_5)+x_6\cdot Prob(X=x_6)$$

$$E[X]=1\cdot(\frac{1}{6})+2\cdot(\frac{1}{6})+3\cdot(\frac{1}{6})+4\cdot(\frac{1}{6})+5\cdot(\frac{1}{6})+6\cdot(\frac{1}{6})$$

$$E[X]=\frac{7}{2}=3.5$$

Outcome	Value x_n	$Prob(X = x_n)$
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x_2	2	1/6
x_3	3	1/6
x_4	4	1/6
x_5	5	1/6
x_6	6	1/6

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

1. Average Value

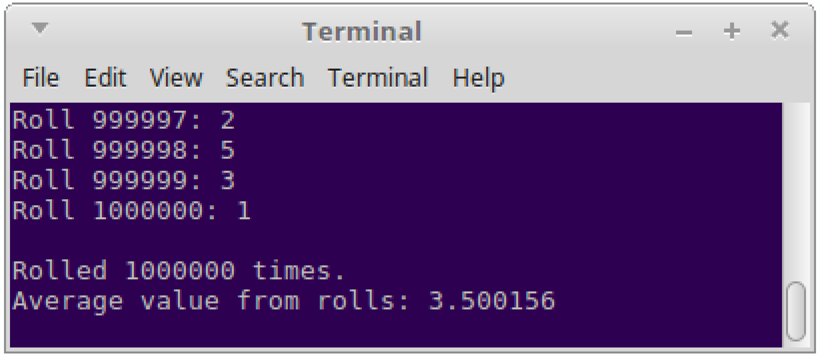
Now let's calculate the average value.

$$E[X]=x_1 \cdot Prob(X=x_1)+x_2 \cdot Prob(X=x_2)+x_3 \cdot Prob(X=x_3)+x_4 \cdot Prob(X=x_4)+x_5 \cdot Prob(X=x_5)+x_6 \cdot Prob(X=x_6)$$

$$E[X]=1 \cdot (\frac{1}{6})+2 \cdot (\frac{1}{6})+3 \cdot (\frac{1}{6})+4 \cdot (\frac{1}{6})+5 \cdot (\frac{1}{6})+6 \cdot (\frac{1}{6})$$

$$E[X]=\frac{7}{2}=3.5$$

We can compare it to the simulation and see that they are close to the same value.



Notes

Expected Value
E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

Expectation in Bernoulli Trials

2. Expectation in Bernoulli Trials

Theorem 1: Suppose an experiment consists of the independent repetition of a trial *n* times, and the probability of that trial's individual success is *p* each time it is performed. If *X* denotes the number of successful trials in this experiment, then

E[X] = n * p

From Discrete Mathematics, Ensley & Crawley, page 469

Notes

Expected Value
E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

Expected Value in a Bernoulli Trial
E[X] = n * p

2. Expectation in Bernoulli Trials

As another example, let's use the die problem, but in this case we are going to consider getting a 1 as a "success" and anything else as a "failure". Our outcomes { x_1, …, x_n } will denote the amount of successes we receive.

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

Expected Value in a Bernoulli Trial

E[X] = n * p

2. Expectation in Bernoulli Trials

For the die trial, let's say we are going to roll the die 5 times. We want to see the expected value.

$$E[X]=x_1\cdot Prob(X=x_1)+x_2\cdot Prob(X=x_2)+x_3\cdot Prob(X=x_3)+x_4\cdot Prob(X=x_4)+x_5\cdot Prob(X=x_5)+x_6\cdot Prob(X=x_6)$$

Outcome	Value x_n	Prob(X = x_n)	
0 successes	0	**C(5, 0) * $(1/6)^0$ * $(5/6)^5$**	
1 success	1	**C(5, 1) * $(1/6)^1$ * $(5/6)^4$**	
2 successes	2	**C(5, 2) * $(1/6)^2$ * $(5/6)^3$**	
3 successes	3	**C(5, 3) * $(1/6)^3$ * $(5/6)^2$**	
4 successes	4	**C(5, 4) * $(1/6)^4$ * $(5/6)^1$**	
5 successes	5	**C(5, 5) * $(1/6)^5$ * $(5/6)^0$**	

Notes

Expected Value
E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

Expected Value in a Bernoulli Trial
E[X] = n * p

Bernoulli trial
n: # of trials
p: Probability of one success
k: Amount of successes

$$C(n,k)\cdot p^k\cdot(1-p)^{n-k}$$

2. Expectation in Bernoulli Trials

For the die trial, let's say we are going to roll the die 5 times. We want to see the expected value.

$$E[X]=x_1\cdot Prob(X=x_1)+x_2\cdot Prob(X=x_2)+x_3\cdot Prob(X=x_3)+x_4\cdot Prob(X=x_4)+x_5\cdot Prob(X=x_5)+x_6\cdot Prob(X=x_6)$$

$$E[X]=0\cdot(\frac{3125}{7776})+1\cdot(\frac{3125}{7776})+2\cdot(\frac{625}{3888})+3\cdot(\frac{125}{3888})+4\cdot(\frac{25}{7776})+5\cdot(\frac{1}{7776})$$

$$E[X]=\frac{5}{6}$$

Outcome	Value x_n	Prob(X = x_n)	
0 successes	0	**C(5, 0) * $(1/6)^0$ * $(5/6)^5$**	= 3125/7776
1 success	1	**C(5, 1) * $(1/6)^1$ * $(5/6)^4$**	= 3125/7776
2 successes	2	**C(5, 2) * $(1/6)^2$ * $(5/6)^3$**	= 625/3888
3 successes	3	**C(5, 3) * $(1/6)^3$ * $(5/6)^2$**	= 125/3888
4 successes	4	**C(5, 4) * $(1/6)^4$ * $(5/6)^1$**	= 25/7776
5 successes	5	**C(5, 5) * $(1/6)^5$ * $(5/6)^0$**	= 1/7776

Notes

Expected Value
E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

Expected Value in a Bernoulli Trial
E[X] = n * p

Bernoulli trial
n: # of trials
p: Probability of one success
k: Amount of successes

$$C(n,k)\cdot p^k\cdot(1-p)^{n-k}$$

2. Expectation in Bernoulli Trials

For the die trial, let's say we are going to roll the die 5 times. We want to see the expected value.

$$E[X]=x_1\cdot Prob(X=x_1)+x_2\cdot Prob(X=x_2)+x_3\cdot Prob(X=x_3)+x_4\cdot Prob(X=x_4)+x_5\cdot Prob(X=x_5)+x_6\cdot Prob(X=x_6)$$

$$E[X]=0\cdot(\frac{3125}{7776})+1\cdot(\frac{3125}{7776})+2\cdot(\frac{625}{3888})+3\cdot(\frac{125}{3888})+4\cdot(\frac{25}{7776})+5\cdot(\frac{1}{7776})$$

$$E[X]=\frac{5}{6}$$

If we calculate it the long way like this, we get 5/6. We can also use the theorem:

$$E[X]=n\cdot p$$

$$E[X]=5\cdot(\frac{1}{6})$$

$$E[X]=\frac{5}{6}$$

(That was easier!)

Notes

Expected Value

E[X] = x1 * Prob(X = x1) + …
+ xn * Prob(X = xn)

Expected Value in a Bernoulli Trial

E[X] = n * p

Bernoulli trial

n: # of trials
p: Probability of one success
k: Amount of successes

$$C(n,k)\cdot p^k\cdot(1-p)^{n-k}$$

Conclusion

Now we've covered the major parts of Chapter 6 – Finding probability, using the Sum and Product rules, and finding Expected Values.