Proofs: Exam Review

Cheat sheet

Cheat Sheet

1. asdf

By Rachel Singh, last updated October 5, 2018

Functions and Function Properties

Be familiar with the following:

- asdf

Practice problems

1. Prove with a direct proof:
   If n is even, then $n + 8$ is even.

2. Prove with a direct proof:
   If n is even, then the product of n and its successor is even.

3. Prove with a direct proof:
   For all integers g and h, if g is **odd** and h is **odd**, then gh is **odd**.

4. Prove with a proof by contrapositive:
   For every $n \in \mathbb{Z}$, if n^2 is even, then n is even.

5. Prove with a proof by contrapositive:
   For every $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

6. Prove with induction:
   Show that the sequence defined by the recursive formula ...

$$a_n = a_{n-1} + 2, \text{ with } a_1 = 1$$

... is equivalently described by the closed formula ...

$$a_n = 2n - 1$$

Steps:
(1) Show that both formulas are equivalent for a_1.
(2) Find the equation for a_{m-1} using the closed formula.
(3) Plug in the equation for a_{m-1} back into the recursive formula and simplify.

7. Prove with induction:
   Show that the sequence defined by the **recursive formula**

$$a_k = a_{k-1} + 4; a_1 = 1$$

for $k \geq 2$ is equivalently described by the **closed formula**

$$a_n = 4n - 3$$

8. Prove with induction: Prove that for each $n \geq 1$,

$$\sum_{i=1}^{n} (2i - 1) = n^2$$

Steps:
(1) Show that they match for $n = 1$, $n = 2$, and $n = 3$.
(2) Rewrite the summation from $i = 1$ to n as equal to the sum from $i = 0$ to $n - 1$, plus the final term at $i = n$.
(3) Find an equation for $\sum_{i=1}^{n-1}$ via the original proposition.
(4) Plug $\sum_{i=1}^{n-1}$ into the equation from step (2) and simplify.

9. Prove with induction:
   As part of the proof, verify the statement for $n = 1$, $n = 2$, and $n = 3$.
   Prove that

$$\sum_{i=1}^{n} (2i + 4) = n^2 + 5n$$

for each $n \geq 1$.

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Answer key

1. If n is even, then $n + 8$ is even.
   $n = 2k$, $\quad n + 8 =¿\ 2k + 8 \quad \boxed{= 2(k + 4)}$

2. If n is even, then the product of n and its successor is even.
   $n = 2k$, $\quad n(n + 1) =¿\ (2k)(2k + 1) \quad = 4k^2 + 2k$
   $\boxed{= 2(2k^2 + k)}$

3. For all integers g and h, if g is **odd** and h is **odd**, then gh is **odd**.
   1. $g = 2k + 1$, $h = 2j + 1$
   2. $gh \equiv (2k + 1)(2j + 1)$
   3. $(2k + 1)(2j + 1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$

   $\boxed{2(2kj + k + j) + 1 \text{ is an odd integer.}}$

4. If n^2 is even, then n is even.
   Contrapositive: If n is not even, then n^2 is not even.
   $n = 2k + 1$, then find n^2 through this.
   $n^2 \equiv (2k + 1)^2 \quad = 4k^2 + 4k + 1 \quad \boxed{= 2(2k^2 + 2k) + 1}$

5. If n^2 is odd, then n is odd.
   Contrapositive: If n is not odd, then n^2 is not odd.
   $n = 2k$, then find n^2 through this.
   $n^2 \equiv (2k)^2 \quad = 4k^2 \quad \boxed{= 2(2k^2)}$

6. Recursive formula: $a_n = a_{n-1} + 2$ with $a_1 = 1$.
   Closed formula: $a_n = 2n - 1$

   Prove for a_1 matches for each.
   Recursive: $a_1 = 1 \quad$ Closed: $a_1 = 2(1) - 1 = 1$ ✓

   Find a formula for a_{n-1} using the closed formula:
   $a_n = 2n - 1 \quad a_{n-1} = 2(n - 1) - 1 \quad a_{n-1} = 2n - 3$

   Plug into recursive formula:
   $a_n = a_{n-1} + 2 \quad a_n = 2n - 3 + 2 \quad \boxed{a_n = 2n - 1}$
   This matches the original proposition.

7. Recursive formula: $a_k = a_{k-1} + 4; a_1 = 1$
   Closed formula: $a_n = 4n - 3$

   Prove for a_1 matches for each.
   Recursive: $a_1 = 1$ Closed: $a_1 = 4(1) - 3 = 1$ ✓

   Find a formula for a_{n-1} using the closed formula:
   $a_n = 4n - 3$ $a_{n-1} = 4(n-1) - 3$ $a_{n-1} = 4n - 7$

   Plug into recursive formula:
   $a_k = a_{k-1} + 4$ (changing k to n) $a_n = 4n - 7 + 4$
   $\boxed{a_n = 4n - 3}$ This matches the original proposition.

8.

$$\sum_{i=1}^{n}(2i-1) = n^2$$

Show it checks out for $i=1$, $i=2$, and $i=3$:

n	Sum	Closed formula	
1	$\sum_{i=1}^{1}(2i-1) = (2(1)-1) = 1$	$n^2 = 1^2 = 1$	✓
2	$\sum_{i=1}^{2}(2i-1) = (2(1)-1)+(2(2)-1) = 1+3 = 4$	$n^2 = 2^2 = 4$	✓
3	$\sum_{i=1}^{3}(2i-1) = (2(1)-1)+(2(2)-1)+(2(3)-1) = 1+3+5 = 9$	$n^2 = 3^2 = 9$	✓

Rewrite sum formula:

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1)$$

Find a formula for $\sum_{i=1}^{n-1}(2i-1)$ using the original proposition:

$$\sum_{i=1}^{n}(2i-1) = n^2$$

$$\sum_{i=1}^{n-1}(2i-1) = (n-1)^2$$

$$\sum_{i=1}^{n-1}(2i-1) = n^2 - 2n + 1$$

Plug into the sum formula we made:

$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1)$

$\sum_{i=1}^{n}(2i-1) = n^2 - 2n + 1 + (2n-1)$

$\boxed{\sum_{i=1}^{n}(2i-1) = n^2}$ Matches the original proposition.

9.

$$\sum_{i=1}^{n}(2i+4) = n^2 + 5n$$

Show it checks out for $i = 1$, $i = 2$, and $i = 3$:

i	Sum	Closed formula	
1	$\sum_{i=1}^{n}(2i+4) = (2(1)+4) = 6$	$n^2 + 5n = 1^2 + 5(1) = 6$	✓
2	$\sum_{i=1}^{n}(2i+4) = (2(1)+4)+(2(2)+4) = 6+8 = 14$	$n^2 + 5n = 2^2 + 5(2) = 14$	✓
3	$\sum_{i=1}^{n}(2i+4) = (2(1)+4)+(2(2)+4)+(2(3)+4) = 6+8+10 = 24$	$n^2 + 5n = 3^2 + 5(3) = 24$	✓

Rewrite sum formula:

$$\sum_{i=1}^{n}(2i+4) = \sum_{i=1}^{n-1}(2i+4) + 2n + 4$$

Find a formula for $\sum_{i=1}^{n-1}(2i-1)$ using the original proposition:

$$\sum_{i=1}^{n}(2i+4) = n^2 + 5n$$

$\sum_{i=1}^{n-1}(2i+4) = (n-1)^2 + 5(n-1)$
$\sum_{i=1}^{n-1}(2i+4) = n^2 - 2n + 1 + 5n - 5$
$\sum_{i=1}^{n-1}(2i+4) = n^2 + 3n - 4$

Plug into the sum formula we made:

$$\sum_{i=1}^{n}(2i+4) = \sum_{i=1}^{n-1}(2i+4) + 2n + 4$$

$\sum_{i=1}^{n}(2i+4) = n^2 + 3n - 4 + 2n + 4$
$\boxed{\sum_{i=1}^{n}(2i+4) = n^2 + 5n}$ Matches the original proposition.