Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group**.

Names:

Proofs: Direct Proofs

Turning statements to implications

Implications

This time we're exploring mathematical writing and getting introduced to proofs. To work with a statement, we turn it into an implication that we can work with mathematically.

Example: For every positive even integer n, $n + 1$ is odd.
Changing to an "if, then" statement, we can form:
If a positive integer n is even, then $n + 1$ is odd.

By Rachel Singh, last updated October 1, 2018

Question 1

Rewrite the following statements as "if, then" statements. They don't need to be *true* statements, we will talk about disproving statements next.

a. When a positive integer n is odd, then $n + 1$ is even.

b. All squares have four equal sides.

> **Hint**
> "If s is a square..."

c. All prime numbers are odd.

> **Hint**
> Remember that a proposition or implication doesn't necessarily have to be the truth; we can determine if it is actually a true or false statement later on.

Counter-examples

Counter-examples

A **counter-example** is a way to disprove a proposition. For implications, if we can keep the **hypothesis** true, and find a scenario where the **conclusion** ends up being false, then we can disprove a statement.

Example: *"If n is a prime number, then n is odd."*
Hypothesis: n is a prime number.
Conclusion: n is odd.

Can we find any examples of prime numbers that aren't odd? We would need something where the hypothesis is true (n is a prime number) and the conclusion is false (n is not odd.)

So yes – 2 is a prime number, and it is not odd. So this statement is false. We have disproven it with a counter-example.

Question 2
Disprove the following statements by coming up with a counter-example:

a. For every even integer n, $n + 1$ is also even.

b. If n is a non-negative integer, then $n! > n$, where $n!$ is n-factorial.

Even, Odd, and Divisibility

In Chapter 2, we will be working with the concept of even and odd numbers a lot, and working out proofs relating to these concepts. But, how do you actually specify that some number is even or odd symbolically?

"An integer is even if it is evenly divisible by two and odd if it is not even."
[...]
"A formal definition of an even number is that it is an integer of the form $n = 2k$, where k is an integer; it can then be shown that an odd number is an integer of the form $n = 2k + 1$." [a]

[a] From https://en.wikipedia.org/wiki/Parity_(mathematics)

Question 3
Identify the following numbers as either even or odd, by writing it as either 2 times some other integer, or 2 times some other integer plus 1.

Example: $21 = 2 \cdot 10 + 1$

a. 7

b. 9

c. 15

d. 8

e. 16

f. 20

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Divisibility

We looked at the definitions for an even and odd number. Here's one more - divisibility!

An integer n is divisible by 4 if it is the result of 4 times some other integer. Symbolically, $n = 4k$.

We can use this definition for divisible by any number, as we need.

Question 4

Using the definitions of even, odd, and divisible by *some integer*, prove that the following statements are true.

- Even items must be of the form 2(some integer).
- Odd items must be of the form 2(some integer) + 1.
- Items divisible by k must be of the form k(some integer).

Example: "12 is even" - rewrite as 2(6).

a. 100 is even.

b. 13 is odd.

c. -13 is odd.

d. 20 is divisible by 5.

e. 20 is divisible by 4.

f. $6n$ is even.

g. $8n^2 + 8n + 4$ is divisible by 4.

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Closure properties of integers

The set of all integers is written as $\mathbb{Z}$.

> *"A set has closure under an operation if performance of that operation on members of the set always produces a member of the same set; in this case we also say that the set is closed under the operation."* [a]

- If you add two integers, the result is also an integer
- If you subtract two integers, the result is also an integer
- If you multiply two integers, the result is also an integer
- If you divide two integers, the result **may not be an integer**

We can use these properties in our proofs, by remembering that if two integers k and j are added, the result $k + j$ is also an integer.

[a] From https://en.wikipedia.org/wiki/Closure_(mathematics)

Question 5
Identify if the result of the following operations belong to the set of integers $\mathbb{Z}$. Write the result as either $\in \mathbb{Z}$ or $\notin \mathbb{Z}$.

a. 2 + 8

b. 12 - 4

c. 5 * 3

d. 6 / 3

e. 5 / 2

Types of Proofs

Direct Proof

Example proof: "The result of summing any odd integer with any even integer is an odd integer."

We should first write this in mathematical language. Let's define our numbers: an even number and an odd number. But how do we write these symbolically?

odd integer	even integer
$x = 2k + 1$	$y = 2j$

An even integer is technically a number that is **evenly divisible by 2**. If we re-word this, we can say that an even integer is "some *other* integer times 2." (Note here that y is our even integer, and it is 2 times some other integer j.)

We know what an even integer is... and an odd integer is just one more than an even number. In this case, we define our odd integer x, as 2 times *some other integer* k, plus 1. (Again note that we're using a different variable for x! j for y and k for x... Make sure to not re-use the same variable.)

With these numbers defined, we can start working mathematically, translating our original statement into symbols.

$x + y$	An odd number plus an even number...
$x + y = (2k + 1) + (2j)$	Adding definitions of odd and even...
$x + y = 2(k + j) + 1$	Rewriting to the definition of an odd #...

$(k + j)$ is still an integer, and the definition of an odd integer is 2 times *some integer* plus 1, proving our statement.

Question 6

Prove the following statements.

a. For all integers $n > 0$, if n is even, then n^2 is also even.

> **Hint**
> Begin with n is even, $n = 2k$, then simplifying 2 to try to get the definition of an even number, $2 \times$ some integer.

b. For all integers $n > 0$, if n is odd, then $n^2 + n$ is even.

> **Hint**
> Begin with n is odd, $n = 2k+1$, then simplifying $(2k+1)^2+(2k+1)$ to try to get the definition of an even number, $2 \times$ some integer.