Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group.**

Names:

Proofs: Proof by Contradiction, Proof by Contrapositive

Review: Direct proofs

With a Direct Proof, you take a statement like, *"The sum of n times $n+1$ is an even number."* and replace the variables with definitions for the relevant type: n becomes $2k+1$ and we can write $(2k+1)(2k+1+1)$. From there, we simplify until we get the form of the definition of an even integer (per this example's statement.)

Question 1

Prove the following statement using a direct proof.

"For all integers a and b, if a is even and b is odd, then $a+b$ is odd."

Start with $a = 2k$ and $b = 2j + 1$.

Hint!

Remember that we're not doing proof by contradiction. We start with equations for a and b, plug these equations into $a + b$, and then simplify and factor into the definition for an odd number.

Proof by Contrapositive

Given some statement, $p \to q$, the contrapositive, $\neg q \to \neg p$ is **logically equivalent**. We can take advantage of this to perform proofs.

Example: Prove that, for all integers n, if n^2 is even, then n is even. [a]

The contrapositive here would be, "if n is not even, then n^2 is not even." Then we can solve as a direct proof...

1. n is odd: $n = 2k + 1$.
2. $n^2 \equiv (2k+1)^2$, then simplify...
3. $(2k+1)^2 = 4k^2 + 4k + 1$, and factor into the form of an odd number...
4. $\boxed{= 2(2k^2 + 2k) + 1.}$

[a] Discrete Mathematics, Ensley and Crawley, p 94-95

Question 2
Prove the following with proof by contrapositive...

"If x and y are two integers whose product is odd, then both must be odd."[1]

1. Write out the hypothesis p, in English:
   Hint: p doesn't include "if"!

2. Write out the conclusion q, in English:

3. Write out the contrapositive $\neg q \rightarrow \neg p$, in English:

4. Give an equation for $\neg q$ (in the contrapositive):

5. Give an equation for $\neg p$ (in the contrapositive):

6. Solve:

[1] From http://zimmer.csufresno.edu/ larryc/proofs/proofs.contrapositive.html

Proof by Contradiction

We can also prove statements by disproving the *negation* of that statement. If we can disprove the negation, when we are proving the original statement.

Example: Prove that, if n^2 is even, then n is even.

Remember that the negation of $p \to q$ is $p \wedge \neg q$. Here, our negation would be n^2 is even and n is not even. Then, we can "translate" this into symbols:

- n^2 is even: $n^2 = 2k$ (some even integer.)
- n is not even: $n = 2j + 1$ (some odd integer.)

As we solve, if we run into a **contradiction**, then we cannot prove the negation, and this shows the proof for the original statement.

$(2j+1)^2 = 2k$	$n = 2j+1$, so squaring it should give us n^2, or some even integer.
$\Rightarrow \quad 4j^2 + 4j + 1 = 2k$	FOILing $(2j+1)^2$
$\Rightarrow \quad 1 = 2k - 4j^2 - 4j$	Move constants to one side
$\Rightarrow \quad 1 = 2(k - 2j^2 - 2j)$	Pull out common factor
$\Rightarrow \quad \boxed{\frac{1}{2} = k - 2j^2 - 2j}$	Divide both sides

Since k and j are both integers, through the closure property of integers (+, -, and $\times$ results in an integer), we can show that $k - 2j^2 - 2j$ results in something that is *not an integer* – this is a contradiction. It shows that our counter-example is **false**, and no counter-example can exist.

Question 3
Prove the following with proof by contradiction...

"If n is an odd integer, then $n^2 + n$ is even." [2]

1. Write out the hypothesis p:

2. Write out the conclusion q:

3. Write out the negation $p \wedge \neg q$:

4. Give an equation for p (in the negation):

5. Give an equation for $\neg q$ (in the negation):

6. Solve:

[2] Discrete Mathematics, Ensley and Crawley, pg 133