Instructions: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. **Work in a team of up to 4 people to complete this exercise.** You can work simultaneously on the problems, or work separate and then check your answers with each other. **Turn in one copy of the exercise per group**.

Names:

Proofs: Mathematical Induction

Sequences

Definition: Recursive formula (aka recurrence relation)
In mathematics, a recurrence relation is an equation that recursively defines a sequence [...] of values, once one or more initial terms are given: each further term of the sequence [...] is defined as a function of the preceding terms. [a]

Definition: Closed formula
A closed formula for a sequence is a formula where each term is described only in relation to its position in the list. [b]

Definition: Sequence notation Sequence notation is where we have some sequence, a, and a_n denotes the element at position n. On a computer, the subscript may be written as `a[n]`.

[a] From https://en.wikipedia.org/wiki/Recurrence_relation
[b] From Discrete Mathematics Mathematical Reasoning and Proof with Puzzles, Patterns, and Games by Douglas E Ensley

 By Rachel Singh, last updated October 1, 2018

Question 1
Write out the first 5 elements of the following equations:

a. The closed formula $a_n = n + 1$
$a_1 =$ $a_2 =$ $a_3 =$ $a_4 =$ $a_5 =$

b. The closed formula $a_n = 2n + 1$

c. The recursive formula $a_1 = 1, a_n = a_{n-1} + 2$
$a_1 = 1$ $a_2 = a_1 + 2 =$ $a_3 = a_2 + 2 =$ *(etc)*

d. The resursive formula $a_1 = 2, a_n = 2a_{n-1} + 1$

Question 2
For each equation, plug $m - 1$ in as n and simplify.

a. $a_n = n + 1$ *Example:* a_{m-1} $= (m-1) + 1$ $= m$

b. $a_n = 2n + 1$

c. $a_1 = 1, a_n = a_{n-1} + 2$

d. $a_1 = 2, a_n = 2a_{n-1} + 1$

Summations

For a sequence of numbers (denoted a_k, where $k >= 1$, we can use the notation

$$\sum_{k=1}^{n} a_k$$

to denote the sum of the first n terms of the sequence. This is called *sigma notation*.

Example: Evaluate the sum $\sum_{k=1}^{3}(2k-1)$.
First, we need to find the elements at $k=1$, $k=2$, and $k=3$:

$k=1$	$k=2$	$k=3$
$a_1 = (2 \cdot 1 - 1) = 1$	$a_2 = (2 \cdot 2 - 1) = 3$	$a_3 = (2 \cdot 3 - 1) = 5$

Then, we can add the values:
$\sum_{k=1}^{3}(2k-1) \quad = a_1 + a_2 + a_3 \quad = 1 + 3 + 5 \quad \boxed{= 9}$

Question 3
Evaluate the following summations.

a. $\sum_{k=1}^{4}(3k)$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Value of $3k$					

Resulting sum:

b. $\sum_{k=1}^{5}(4)$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Value of 4					

Resulting sum:

Recursive / Closed formula equivalence

Example

Show that the sequence defined by the **recursive formula** [a]

$$a_k = a_{k-1} + 4; a_1 = 1$$

for $k \geq 2$ is equivalently described by the **closed formula**

$$a_n = 4n - 3$$

Basis Step: Check a_1 for both formulas.

Recursive: $a_1 = 1$ (provided); Closed: $a_1 = 4(1) - 3 = 1$ ✓OK

Inductive Step: Show that this is true for all values up through $n-1$:

Find an equation for a_{k-1} via the closed formula provided:

Original proposition: $a_n = 4n-3$ and $a_k = a_{k-1}+4; a_1 = 1$ are equivalent. Use $a_n = 4n - 3$ to find a value for a_{k-1}.

1. $a_n = 4n - 3$	The closed formula
2. $a_{k-1} = 4(k - 1) - 3$	Plugging in $k - 1$ into n
3. $a_{k-1} = 4k - 4 - 3$	Simplifying...
4. $a_{k-1} = 4k - 7$	Simplified.

Plug the equation for a_{k-1} into the recursive formula and simplify.

1. $a_k = a_{k-1} + 4$	The recursive formula
2. $a_k = 4k - 7 + 4$	Plugging in $a_{k-1} = 4k - 7$.
3. $\boxed{a_k = 4k - 3}$	Simplified to the original form.

We have manipulated the **recursive formula** to end up with the same **closed formula** as stated in the original proposition, therefore we have shown that they are equivalent.

[a] From Discrete Mathematics, Ensley and Crawley

Question 4
Show that the sequence defined by $a_n = a_{n-1} + 2; a_1 = 5$ for $k \geq 2$ is equivalently described by the closed formula, $a_n = 2n + 3$. [1]

Basis Step: Check a_1 for both formulas.

Inductive Step:

Find the equation for a_{n-1} via the closed formula:

Plug a_{n-1} back into the recursive formula and simplify.

[1] From Discrete Mathematics by Ensley and Crawley

Exponent rules
Power rule: $(a^m)^n = a^{mn}$
Negative exponent rule: $a^{-n} = \frac{1}{a^n}$
Product rule: $a^m \cdot a^n = a^{m+n}$
Quotient rule: $\frac{a^m}{a^n} = a^{m-n}$

Question 5

Show that the sequence defined by $b_k = 4 \cdot b_{k-1} + 3, b_1 = 3$ for $k \geq 2$, is equivalently described by the closed formula $b_n = 2^{2n} - 1$. [2]

Basis Step: Check b_1 for both formulas.

Inductive Step:

Find the equation for b_{k-1} via the closed formula:

Plug b_{k-1} back into the recursive formula and simplify.

[2] From Discrete Mathematics by Ensley and Crawley

Sum / Closed formula equivalence

Example

Use induction to prove the proposition. As part of the proof, verify the statement for $n = 1$, $n = 2$, and $n = 3$. $\sum_{i=1}^{n}(2i-1) = n^2$ for each $n \geq 1$. [a]

Basis step: Show that the proposition is true for 1, 2, and 3.

$i = 1$:	LHS: $\sum_{i=1}^{1}(2i-1) = (2 \cdot 1 - 1) = 1$	RHS: $1^2 = 1$ ✓
$i = 2$:	LHS: $\sum_{i=1}^{2}(2i-1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) = (1) + (3) = 4$	RHS: $2^2 = 4$ ✓ [b]
$i = 3$:	LHS: $\sum_{i=1}^{3}(2i-1) = 1 + 3 + (2 \cdot 3 - 1) = 1 + 3 + 5 = 9$	RHS: $3^2 = 9$ ✓

Inductive Step

The sum is equivalent to the sum up until $n - 1$, plus the final term at $i = n$:

1. $\sum_{i=1}^{n}(2i-1)$ — The original sum.
2. $\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1)$ — The sum is equivalent to the sum from $i = 1$ to $n-1$, plus the final $i = n$.

Find an equation for $\sum_{i=1}^{n-1}$ from the original proposition:

1. $\sum_{i=1}^{n}(2i-1) = n^2$ — Original proposition.
2. $\sum_{i=1}^{n-1}(2i-1) = (n-1)^2$ — Plugging in $n - 1$.
3. $\sum_{i=1}^{n-1}(2i-1) = n^2 - 2n + 1$ — Simplified.

Plug $\sum_{i=1}^{n-1}$ into the equation for the sum made previously:

1. $\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1)$ — Our sum formula.
2. $\sum_{i=1}^{n}(2i-1) = (n^2 - 2n + 1) + (2n-1)$ — plugged in $\sum_{i=1}^{n-1}(2i-1) = n^2 - 2n + 1$.
3. $\boxed{\sum_{i=1}^{n}(2i-1) = n^2}$ — Simplified to the original proposition.

We get the same form as the original proposition, proving our statement.

[a] From Discrete Mathematics by Ensley and Crawley
[b] LHS = left-hand side, RHS = right-hand side

Question 6

Use induction to prove

$$\sum_{i=1}^{n}(2i+4) = n^2 + 5n$$

for each $n \geq 1$.

Basis Step: Show that the proposition is true for 1, 2, and 3.

i **value**	$\sum_{i=1}^{n}(2i+4)$	$n^2 + 5n$
$i = 1$		
$i = 2$		
$i = 3$		

Inductive Step:

Make an equation for the sum that is the sum up to $n-1$, plus the final term:

Find an equation for $\sum_{i=1}^{n-1}$ via the proposition:

Plug $\sum_{i=1}^{n-1}$ into the sum equation and simplify:

Question 7

Use induction to prove that for every positive integer n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Basis Step: Show that the proposition is true for 1, 2, and 3.

i **value**	$\sum_{i=1}^{n}(2i+4)$	n^2+5n
$i=1$		
$i=2$		
$i=3$		

Inductive Step:

Make an equation for the sum that is the sum up to $n-1$, plus the final term:

Find an equation for $\sum_{i=1}^{n-1}$ via the proposition:

Plug $\sum_{i=1}^{n-1}$ into the sum equation and simplify: