Instructions: Work on homework assignments to further familiarize yourself with the topics in the class. The answers are provided for these problems. You can work with other students as desired. Turn in your work on canvas to be given a grade for completion (homework will not be checked for correctness; you need to verify this yourself.)

Upload each homework assignment to its own "dropbox" on Canvas.

This document is not formatted to be written on; do your homework on a separate sheet of paper.

Review: Rules of exponents

- Power Rule: $(a^m)^n = a^{mn}$
- Negative Exponent Rule: $a^{-n} = \frac{1}{a^n}$
- Product Rule: $a^m \cdot a^n = a^{m+n}$
- Quotient Rule: $\frac{a^m}{a^n} = a^{m-n}$

By Rachel Singh, last updated October 1, 2018

Recursive / Closed formula equivalence

Example

Show that the sequence defined by the **recursive formula** [a]

$$a_k = a_{k-1} + 4; a_1 = 1$$

for $k \geq 2$ is equivalently described by the **closed formula**

$$a_n = 4n - 3$$

Basis Step: Check a_1 for both formulas.

Recursive: $a_1 = 1$ (provided); Closed: $a_1 = 4(1) - 3 = 1$ ✓OK

Inductive Step: Show that this is true for all values up through $n-1$:

Find an equation for a_{k-1} via the closed formula provided:

Original proposition: $a_n = 4n-3$ and $a_k = a_{k-1}+4; a_1 = 1$ are equivalent. Use $a_n = 4n - 3$ to find a value for a_{k-1}.

1. $a_n = 4n - 3$	The closed formula
2. $a_{k-1} = 4(k-1) - 3$	Plugging in $k-1$ into n
3. $a_{k-1} = 4k - 4 - 3$	Simplifying...
4. $a_{k-1} = 4k - 7$	Simplified.

Plug the equation for a_{k-1} into the recursive formula and simplify.

1. $a_k = a_{k-1} + 4$	The recursive formula
2. $a_k = 4k - 7 + 4$	Plugging in $a_{k-1} = 4k - 7$.
3. $\boxed{a_k = 4k - 3}$	Simplified to the original form.

We have manipulated the **recursive formula** to end up with the same **closed formula** as stated in the original proposition, therefore we have shown that they are equivalent.

[a] From Discrete Mathematics, Ensley and Crawley

Example

Use induction to prove the proposition. As part of the proof, verify the statement for $n = 1$, $n = 2$, and $n = 3$. $\sum_{i=1}^{n}(2i-1) = n^2$ for each $n \geq 1$. [a]

Basis step: Show that the proposition is true for 1, 2, and 3.

$i = 1$:	LHS: $\sum_{i=1}^{1}(2i-1) = (2 \cdot 1 - 1) = 1$	RHS: $1^2 = 1$ ✓
$i = 2$:	LHS: $\sum_{i=1}^{2}(2i-1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) = (1) + (3) = 4$	RHS: $2^2 = 4$ ✓ [b]
$i = 3$:	LHS: $\sum_{i=1}^{3}(2i-1) = 1 + 3 + (2 \cdot 3 - 1) = 1 + 3 + 5 = 9$	RHS: $3^2 = 9$ ✓

Inductive Step

The sum is equivalent to the sum up until $n-1$, plus the final term at $i = n$:

1. $\sum_{i=1}^{n}(2i-1)$ — The original sum.
2. $\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1)$ — The sum is equivalent to the sum from $i = 1$ to $n-1$, plus the final $i = n$.

Find an equation for $\sum_{i=1}^{n-1}$ from the original proposition:

1. $\sum_{i=1}^{n}(2i-1) = n^2$ — Original proposition.
2. $\sum_{i=1}^{n-1}(2i-1) = (n-1)^2$ — Plugging in $n-1$.
3. $\sum_{i=1}^{n-1}(2i-1) = n^2 - 2n + 1$ — Simplified.

Plug $\sum_{i=1}^{n-1}$ into the equation for the sum made previously:

1. $\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2n-1)$ — Our sum formula.
2. $\sum_{i=1}^{n}(2i-1) = (n^2 - 2n + 1) + (2n-1)$ — plugged in $\sum_{i=1}^{n-1}(2i-1) = n^2 - 2n + 1$.
3. $\boxed{\sum_{i=1}^{n}(2i-1) = n^2}$ — Simplified to the original proposition.

We get the same form as the original proposition, proving our statement.

[a] From Discrete Mathematics by Ensley and Crawley
[b] LHS = left-hand side, RHS = right-hand side

1. Use induction to prove that each sum is equivalent to each formula for each $n \geq 1$ [1]

   a.
   $$\sum_{i=1}^{n}(2i-1) = n^2$$

   b.
   $$\sum_{i=1}^{n}(2i+4) = n^2 + 5n$$

   c.
   $$\sum_{i=1}^{n}(2^i - 1) = 2^{n+1} - n - 2$$

2. Use induction to prove that each pair of recursive formulas and closed formulas are equivalent to each other. [2]

   a. Recursive: $a_k = a_{k-1} + 4$, $a_1 = 1$ for $k \geq 2$
      Closed: $a_n = 4n - 3$.

   b. Recursive: $a_k = a_{k-1} + (k+4)$, $a_1 = 5$ for $k \geq 2$
      Closed: $a_n = \frac{n(n+9)}{2}$.

   c. Recursive: $a_k = a_{k-1} + k^2$, $a_1 = 1$ for $k \geq 2$
      Closed: $a_n = \frac{n(n+1)(2n+1)}{6}$.

   d. Recursive: $a_k = 2a_{k-1} + 1$, $a_1 = 1$ for $k \geq 2$
      Closed: $a_n = 2^n - 1$.

[1] Discrete Mathematics, Ensley and Crawley, p 122
[2] Discrete Mathematics, Ensley and Crawley, p 121

Mathematical Induction - Answer key

1. a.

$$\sum_{k=1}^{n}(2k-1) = n^2$$

Basis Step: Check for $n = 1$, $n = 2$, $n = 3$:

n	Summation	Equation	
$n = 1$	$\sum_{k=1}^{1}(2k-1) = (2\cdot 1 - 1) = 1$	$1^2 = 1$	✓
$n = 2$	$\sum_{k=1}^{2}(2k-1) = (2\cdot 1-1)+(2\cdot 2-1) = 1+3 = 4$	$2^2 = 4$	✓
$n = 3$	$\sum_{k=1}^{3}(2k-1) = (2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3-1) = 1+3+5 = 9$	$3^2 = 9$	✓

Inductive Steps:

– Rewrite the sum as the sum of all items from $k = 1$ to $n-1$, plus the final term $2n-1$.

$$\sum_{k=1}^{n}(2k-1) \equiv \sum_{k=1}^{n-1}(2k-1) + (2n-1)$$

– We need an equation for $\sum_{k=1}^{n-1}(2k-1)$, so we use the original proposition here.
$\sum_{k=1}^{n}(2k-1) = n^2$ ... $\sum_{k=1}^{n-1}(2k-1) = (n-1)^2$ ...

$$\sum_{k=1}^{n-1}(2k-1) = n^2 - 2n + 1$$

– Plug the equation for $\sum_{k=1}^{n-1}(2k-1)$ back into the equation for $\sum_{k=1}^{n}(2k-1)$ and simplify:
$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n-1}(2k-1) + (2n-1)$
$\sum_{k=1}^{n}(2k-1) = n^2 - 2n + 1 + (2n-1)$
$\sum_{k=1}^{n}(2k-1) = n^2$

$$\sum_{k=1}^{n}(2k-1) = n^2$$

This is the same form as the original proposition. Therefore, we have proven it.

b.

$$\sum_{i=1}^{n}(2i+4) = n^2 + 5n$$

Basis Step:

n	Summation	Equation	
$n=1$	$\sum_{k=1}^{1}(2i+4) = (2\cdot 1+4) = 6$	$1^2 + 5(1) = 6$	✓
$n=2$	$\sum_{k=1}^{2}(2i+4) = (2\cdot 1+4) + (2\cdot 2+4) = 6+8 = 14$	$2^2 + 5(2) = 14$	✓
$n=3$	$\sum_{k=1}^{3}(2i+4) = (2\cdot 1+4)+(2\cdot 2+4) + (2\cdot 3+4) = 6+8+10 = 24$	$3^2 + 5(3) = 24$	✓

Inductive Steps:

- Rewrite sum:

$$\sum_{k=1}^{n}(2i+4) \equiv \sum_{k=1}^{n-1}(2i+4) + (2n+4)$$

- Find equation for sum from 1 to $n-1$:
  $\sum_{i=1}^{n-1}(2i+4) = (n-1)^2 + 5(n-1)$
  $\sum_{i=1}^{n-1}(2i+4) = n^2 - 2n + 1 + 5n - 5$
  $\sum_{i=1}^{n-1}(2i+4) = n^2 + 3n - 4$

- Plug into sum equation:
  $\sum_{k=1}^{n}(2i+4) = \sum_{k=1}^{n-1}(2i+4) + (2n+4)$
  $\sum_{k=1}^{n}(2i+4) = n^2 + 3n - 4 + (2n+4)$
  $\sum_{k=1}^{n}(2i+4) = n^2 + 5n$

$$\boxed{\sum_{k=1}^{n}(2i+4) = n^2 + 5n}$$

We got it back to the original form, so we have proven the proposition.

c.

$$\sum_{i=1}^{n}(2^i - 1) = 2^{n+1} - n - 2$$

Basis Step:

n	Summation	Equation	
$n = 1$	$\sum_{k=1}^{1}(2^i - 1) = (2^1 - 1) = 1$	$2^{1+1} - 1 - 2 = 4 - 1 - 2 = 1$	✓
$n = 2$	$\sum_{k=1}^{2}(2^i - 1) = (2^1 - 1) + (2^2 - 1) = 1 + 3 = 4$	$2^{2+1} - 2 - 2 = 8 - 2 - 2 = 4$	✓
$n = 3$	$\sum_{k=1}^{3}(2^i - 1) = (2^1 - 1) + (2^2 - 1) + (2^3 - 1) = 1 + 3 + 7 = 11$	$2^{3+1} - 3 - 2 = 16 - 3 - 2 = 11$	✓

Inductive Steps:

- Rewrite sum:

$$\sum_{i=1}^{n}(2^i - 1) \equiv \sum_{i=1}^{n-1}(2^i - 1) + (2^n - 1)$$

- Find sum from 1 to $n - 1$:
  $\sum_{i=1}^{n-1}(2^i - 1) = 2^{n-1+1} - (n - 1) - 2$
  $\sum_{i=1}^{n-1}(2^i - 1) = 2^n - n + 1 - 2$
  $\sum_{i=1}^{n-1}(2^i - 1) = 2^n - n - 1$

- Plug into sum equation:
  $\sum_{i=1}^{n}(2^i - 1) = \sum_{i=1}^{n-1}(2^i - 1) + (2^n - 1)$
  $\sum_{i=1}^{n}(2^i - 1) = 2^n - n - 1 + (2^n - 1)$
  $\sum_{i=1}^{n}(2^i - 1) = 2^n + 2^n - n - 1 - 1$
  $\sum_{i=1}^{n}(2^i - 1) = 2(2^n) - n - 2$
  $\sum_{i=1}^{n}(2^i - 1) = (2^1)(2^n) - n - 2$
  $\sum_{i=1}^{n}(2^i - 1) = 2^{n+1} - n - 2$

$$\sum_{i=1}^{n}(2^i - 1) = 2^{n+1} - n - 2$$

For this one, I had to add two items with like terms, $2^n + 2^n$. If we had $a + a$, the result would be $2a$, so here instead the result is $2(2^n)$. Then, using the product rule, that gives us $2^1 \cdot 2^n$ and then 2^{n+1}.

2. a. Recursive: $a_k = a_{k-1} + 4$, $a_1 = 1$ for $k \geq 2$
Closed: $a_n = 4n - 3$.

Basis step:

Recursive: $a_1 = 1$, Closed: $a_1 = 1$ ✓

Inductive steps:

Find equation for a_{k-1}:
$a_n = 4n - 3$ $a_{k-1} = 4(k-1) - 3$, $a_{k-1} = 4k - 7$

Plug into recursive:
$a_k = a_{k-1} + 4$, $a_k = 4k - 7 + 4$, $\boxed{a_k = 4k - 3}$

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b. Recursive: $a_k = a_{k-1} + (k+4)$, $a_1 = 5$ for $k \geq 2$
Closed: $a_n = \frac{n(n+9)}{2}$.

Basis step:

Recursive: $a_1 = 5$, Closed: $a_1 = \frac{1(10)}{2} = 5$ ✓

Inductive steps:

Find equation for a_{k-1}:
$a_n = \frac{n(n+9)}{2}$ $a_{k-1} = \frac{(k-1)(k-1+9)}{2}$, $a_{k-1} = \frac{(k-1)(k+8)}{2}$

$a_{k-1} = \frac{k^2+7k-8}{2}$

Plug into recursive:
$a_k = a_{k-1} + (k+4)$, $a_k = \frac{k^2-7k-8}{2} + (k+4)$, $a_k = \frac{k^2+7k-8}{2} + \frac{2(k+4)}{2}$

$a_k = \frac{k^2+7k-8+2k+8}{2}$ $a_k = \frac{k^2+9k}{2}$ $\boxed{a_k = \frac{k(k+9)}{2}}$

c. Recursive: $a_k = a_{k-1} + k^2$, $a_1 = 1$ for $k \geq 2$
Closed: $a_n = \frac{n(n+1)(2n+1)}{6}$.

Basis step:

Recursive: $a_1 = 1$, Closed: $a_1 = \frac{1(1+1)(2\cdot1+1)}{6} = \frac{1(2)(3)}{6} = 1$ ✓

Inductive steps:

Find equation for a_{k-1}:
$a_n = \frac{n(n+1)(2n+1)}{6}$ $a_{k-1} = \frac{(k-1)(k-1+1)(2(k-1)+1)}{6}$,

$a_{k-1} = \frac{(k-1)(k)(2k-1)}{6}$ $a_{k-1} = \frac{(k^2-k)(2k-1)}{6}$

$a_{k-1} = \frac{2k^3-k^2-2k^2+k}{6}$ $a_{k-1} = \frac{2k^3-3k^2+k}{6}$

Plug into recursive:
$a_k = a_{k-1} + k^2$, $a_k = \frac{2k^3-3k^2+k}{6} + k^2$, $a_k = \frac{2k^3-3k^2+k}{6} + \frac{6k^2}{6}$,

$a_k = \frac{2k^3-3k^2+k+6k^2}{6}$, $a_k = \frac{2k^3+3k^2+k}{6}$, $a_k = \frac{k(2k^2+3k^1+1)}{6}$,

$a_k = \frac{k(2k^2+3k+1)}{6}$, $\boxed{a_k = \frac{k(2k+1)(k+1)}{6}}$

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d. Recursive: $a_k = 2a_{k-1} + 1$, $a_1 = 1$ for $k \geq 2$
Closed: $a_n = 2^n - 1$.

NOTE: Product Rule: $a^m \cdot a^n = a^{m+n}$

Basis step:

Recursive: $a_1 = 1$, Closed: $a_1 = 2^1 - 1 = 1$ ✓

Inductive steps:

Find equation for a_{k-1}:
$a_n = 2^n - 1$ $a_{k-1} =$

Plug into recursive:
$a_k = 2a_{k-1} + 1$, $a_k = 2(2^{k-1} - 1) + 1$,
$a_k = 2^1 \cdot 2^k \cdot 2^{-1} - 2 + 1$, $a_k = \frac{2}{2} \cdot 2^k - 1$,
$\boxed{a_k = \cdot 2^k - 1}$