Proofs: Introduction

Written by Rachel Singh, last updated Oct 1, 2018

About

When making a proposition, how would you prove that what you say is true for all cases?

When writing a program, how are you sure it does what you expect it to do?

In this part we will look at proofs.

Topics

1. Turning statements into implications
2. Counter-examples
3. Even, Odd, and Divisibility
4. The Closer Property of Integers
5. Direct Proofs

Turning Statements into Implications

1. Turning Statements into Implications

This time we're exploring mathematical writing and getting introduced to proofs. To work with a statement, we turn it into an implication that we can work with mathematically.

It will be easier to come up with a counter-example to a statement if we can put it in an “if, then” framework first.

Notes

Implication:
$p \rightarrow q$

1. Turning Statements into Implications

Example:
For every positive even integer n, $n+1$ is odd.

Notes

Implication:
$p \rightarrow q$

p: hypothesis
q: conclusion

1. Turning Statements into Implications

Example:
For every positive even integer *n*, *n+1* is odd.

Hypothesis — Conclusion

"If a positive integer ***n*** is even,
then ***n+1*** is odd."

Notes

Implication:
$p \to q$

p: hypothesis
q: conclusion

1. Turning Statements into Implications

Example:
For every positive even integer *n*, *n+1* is odd.

"If a positive integer ***n*** is even,
then ***n+1*** is odd."

Given this statement, if we wanted to disprove it, what would we need to come up with?

Notes

Implication:
$p \rightarrow q$

p: hypothesis
q: conclusion

Counter-Examples

2. Counter-Examples

A **counter-example** is a way to disprove a proposition.

For implications, if we can keep the **hypothesis** true, and find a scenario where the **conclusion** ends up being false, then we can disprove a statement.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

2. Counter-Examples

Example:
If n is a prime number, then n is odd.

What is the **hypothesis** and the **conclusion**?

Notes

Implication:
$p \rightarrow q$

Negation:
$p \wedge \neg q$

2. Counter-Examples

Example:
If n is a prime number, then n is odd.

What is the **hypothesis** and the **conclusion**?

Hypothesis: n is a prime number
Conclusion: n is odd

What is the negation of the implication?

Notes

Implication:
$p \to q$

Negation:
$p \land \neg q$

2. Counter-Examples

Example:
If n is a prime number, then n is odd.

What is the **hypothesis** and the **conclusion**?

Hypothesis: n is a prime number
Conclusion: n is odd

What is the negation of the implication?

n is a prime number and n is NOT odd.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \wedge \neg q$

2. Counter-Examples

Example:
If *n* is a prime number, then *n* is odd.

Implication: if *n* is a prime number then *n* is odd.

Negation: *n* is a prime number and *n* is NOT odd.

Can we find a counter-example that makes the negation **true?**

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

2. Counter-Examples

Example:
If *n* is a prime number, then *n* is odd.

Implication: if *n* is a prime number then *n* is odd.

Negation: *n* is a prime number and *n* is NOT odd.

Can we find a counter-example that makes the negation **true?**

2 is a prime number and 2 is NOT odd.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

2. Counter-Examples

Example:
If n is a prime number, then n is odd.

Implication: if n is a prime number then n is odd.

Negation: n is a prime number and n is NOT odd.

2 is a prime number and 2 is NOT odd.

We have found at least one counter-example, so this means that the original implication has been **disproven.**

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Even, Odd, and Divisibility

3. Even, Odd, and Divisibility

In Chapter 2, we will be working with the concept of even and odd numbers a lot, and working out proofs relating to these concepts.

But, how do you actually specify that some number is even or odd *symbolically*?

Notes

Even:

Odd:

Divisible by x:

3. Even, Odd, and Divisibility

"A formal definition of an even number is that it is an integer of the form ***n = 2k***, where ***k*** is an integer;

it can then be shown that an odd number is an integer of the form ***n = 2k + 1***."

From https://en.wikipedia.org/wiki/Parity_(mathematics)

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:

3. Even, Odd, and Divisibility

Example: Rewrite the following using the definition of an even or odd number to "prove" that it is even or odd:

1. 5

2. 6

3. -11

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:

3. Even, Odd, and Divisibility

Example: Rewrite the following using the definition of an even or odd number to "prove" that it is even or odd:

1. 5 ***= 2(2) + 1*** ***It is odd***
2. 6 ***= 2(3)*** ***It is even***
3. -11 ***= 2(-6) + 1*** ***It is odd***

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:

3. Even, Odd, and Divisibility

We looked at the definitions for an even and odd number.
Here's one more – divisibility!

"An integer ***n*** is divisible by ***4*** if it is the result of ***4*** times some other integer. Symbolically, ***n = 4k***."

Rewrite the following number to show that it is divisible by
32

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:
n = xk

3. Even, Odd, and Divisibility

We looked at the definitions for an even and odd number.
Here's one more – divisibility!

"An integer ***n*** is divisible by ***4*** if it is the result of ***4*** times some other integer. Symbolically, ***n = 4k***."

Rewrite the following number to show that it is divisible by
32 ***= 8(4)***

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:
n = xk

The Closer Property of Integers

4. The Closure Property of Integers

"A set has closure under an operation if performance of **that operation on members of the set always produces a member of the same set**; in this case we also say that the set is closed under the operation."

From https://en.wikipedia.org/wiki/Closure_(mathematics)

Notes

4. The Closure Property of Integers

“A set has closure under an operation if performance of **that operation on members of the set always produces a member of the same set**; in this case we also say that the set is closed under the operation."

From https://en.wikipedia.org/wiki/Closure_(mathematics)

- If you add two integers, the result is also an integer
- If you subtract two integers, the result is also an integer
- If you multiply two integers, the result is also an integer
- If you divide two integers, the result **may not be an integer**.

Notes

Direct Proofs

5. Direct Proofs

With a **Direct Proof**, we prove a statement by substituting the variables with our definitions for even, odd, or divisible by. From there, we work out the math and simplify until we get a statement that proves the statement.

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Say we have "if n is even, then n+1 is odd". We would start by coming up with a substitution for n…

$n = 2k$

Then we would substitute it into the n+1 statement, hopefully simplifying it into the definition of some odd number (to match the original proposition).

$n+1$ is odd $\quad$ $(2k) + 1$ is odd $\quad$ **$2k + 1$ is odd**

This is in the form of the definition of an odd integer, so we have proven the statement.

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example Direct Proof:
“The result of summing any odd integer with any even integer is an odd integer."

First, what is the hypothesis and the conclusion?

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

Example Direct Proof:
“The result of summing any odd integer with any even integer is an odd integer."

First, what is the hypothesis and the conclusion?

If x is an odd integer and y is an even integer, then x + y is an odd integer.

I made up variables “x” and “y” here.

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example Direct Proof:
“The result of summing any odd integer with any even integer is an odd integer."

If x is an odd integer and y is an even integer, then x + y is an odd integer.

Now we want to come up with definitions for the variables used in this statement: x and y.

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example Direct Proof:
“The result of summing any odd integer with any even integer is an odd integer."

If x is an odd integer and y is an even integer, then x + y is an odd integer.

Now we want to come up with definitions for the variables used in this statement: x and y.

$$x = 2k + 1 \qquad y = 2j$$

Make sure that, for different variables, you use different “alias” variables as well! Don’t re-use “k”!

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example Direct Proof:
"The result of summing any odd integer with any even integer is an odd integer."

If x is an odd integer and y is an even integer, then x + y is an odd integer.

$$x = 2k + 1 \qquad y = 2j$$

The conclusion talks about x + y, so this is where we start our math – but we substitute x and y with the equations above.

$x + y$ => $2k+1 + 2j$ Now we simplify

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example Direct Proof:
"The result of summing any odd integer with any even integer is an odd integer."

$x + y$ => $2k+1 + 2j$
$2k + 2j + 1$
$2(k+j) + 1$

All we have to do is simplify (if we can) and factor terms as appropriate to try to get to the definition of an odd integer.

An odd integer is 2 times "some integer" plus 1. k+j is "some integer", since adding two integers gives us another integer.

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example 2: If *n* is even and *m* is odd, then *n x m* is even.

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:
n = xk

Example 2: If n is even and m is odd, then $n \times m$ is even.

n x m	**=>**	**(2k)(2j+1)**	**Even number times odd number**
	=	**4kj + 2k**	**Multiply through**
	=	**2(2kj + k)**	**Factor out the 2**

2(2kj + k) is in the form of the definition of an even integer, so therefore we have proven the original statement.

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:
n = xk

5. Direct Proofs

Example 3: If a number is divisible by 6, then it is even.

Notes

Even:
$n = 2k$

Odd:
$n = 2k + 1$

Divisible by x:
$n = xk$

5. Direct Proofs

Example 3: If a number is divisible by 6, then it is even.

n	**= 6k**	**Divisible by 6**
	= 2(3k)	**Factored out the 2**

2(3k) is in the form of the definition of an even integer, so therefore we have proven the original statement.

Notes

Even:
n = 2k

Odd:
n = 2k + 1

Divisible by x:
n = xk

Conclusion

Doing proofs takes practice to learn and remember the steps. Make sure to give yourself time to work through problems and practice, practice, practice!