More methods of Proof: Contradictions and Contrapositives

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About

Sometimes, a direct proof is not a viable way to prove a statement. In these cases, we can do a **proof by contradiction** or a **proof by contrapostive** to prove the proposition.

Topics

1. Proof by Contrapositive
2. Proof by Contradiction
3. Getting them mixed up

Proof by Contrapositive

1. Proof by Contrapositive

For any integer n, if n^2 is even, then n is even.

How would you go about using a direct proof on this statement?

Remember that we start at the hypothesis, give it an equation, and then work with the conclusion to prove the statement.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

1. Proof by Contrapositive

For any integer n, if n^2 is even, then n is even.

Because we have n^2 as the hypothesis, this makes the proof hard to do with a direct proof.

But, we can flip it around.

The contrapositive of an implication is **logically equivalent** to the implication, so we just change the statement!

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

1. Proof by Contrapositive

Original: if n^2 is even, then n is even.
Contrapositive: if n is not even, then n^2 is not even.

This is a statement we can do a direct proof on.

Notes

Implication:
p → q

Negation:
p ∧ ¬q

Contrapositive:
¬q → ¬p

1. Proof by Contrapositive

Original: if n^2 is even, then n is even.
Contrapositive: if n is not even, then n^2 is not even.

This is a statement we can do a direct proof on.

$n = 2k+1$

$$n^2 \Rightarrow (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

$2(2k^2 + 2k) + 1$ (2 times some integer plus 1)
is in the form of the definition of an odd number, so we have proven the statement.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \wedge \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

Proof by Contradiction

2. Proof by Contradiction

If n^2 is even, then n is even.

Another way we can prove this statement is with a proof by contradiction.

In this case, we try to come up with a counter-example, plug it into the statement, and simplify until we come up with some sort of contradiction in the math itself.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

2. Proof by Contradiction

Implication: If n^2 is even, then n is even.
Negation: n^2 is even and n is not even.

We are going to assume that we have found some counter-example, such that n^2 is even and n itself is not even.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

2. Proof by Contradiction

Implication: If n^2 is even, then n is even.
Negation: n^2 is even and n is not even.

In this scenario, we need “alias variables” for both sides.

n^2 is even: 2k **n is not even: 2j + 1**

Notes

Implication:
p → q

Negation:
p ∧ ¬q

Contrapositive:
¬q → ¬p

2. Proof by Contradiction

Implication: If n^2 is even, then n is even.
Negation: n^2 is even and n is not even.

n^2 is even: 2k **n is not even: 2j + 1**

Next, we know that if we times n by itself, we get n^2, so our equation looks like this:

$$(n)(n) = n^2$$

Then we can plug in our "even" and "odd" equations.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

2. Proof by Contradiction

Implication: If n^2 is even, then n is even.
Negation: n^2 is even and n is not even.

n^2 is even: 2k **n is not even: 2j + 1**

$(n)(n) = n^2$
$(2j+1)(2j+1) = 2k$
$4j^2 + 4j + 1 = 2k$
$2(2j^2 + 2j) + 1 = 2k$

We can already see that an odd integer is NOT equal to an even integer, but we can keep working…

Notes

Implication:
$p \to q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \to \neg p$

2. Proof by Contradiction

Implication: If n^2 is even, then n is even.

Negation: n^2 is even and n is not even.

n^2 is even: $2k$　　**n is not even: $2j + 1$**

$$(2j+1)(2j+1) = 2k$$
$$4j^2 + 4j + 1 = 2k$$
$$1 = 2(k - 2j^2 + 2j)$$

We can move the constant to its own side and factor out that 2 from the variables side. Then, when we divide both sides by 2…

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

2. Proof by Contradiction

Implication: If n^2 is even, then n is even.

Negation: n^2 is even and n is not even.

n^2 is even: $2k$ n is not even: $2j + 1$

$$1 = 2(k - 2j^2 + 2j)$$

$$\tfrac{1}{2} = k - 2j^2 + 2j$$

We uncover a fraction! The result of $k - 2j^2 + 2j$ ***cannot*** be a fraction, as these are all integers, and doing multiplication, addition, and subtraction on integers should always result in an integer!

Notes

Implication:
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2. Proof by Contradiction

Implication: If n^2 is even, then n is even.

Negation: n^2 is even and n is not even.

n^2 is even: $2k$ **n is not even: $2j + 1$**

$$
\begin{aligned}
(n)(n) &= n^2 \\
(2j+1)(2j+1) &= 2k \\
4j^2 + 4j + 1 &= 2k \\
2(2j^2 + 2j) + 1 &= 2k \\
1 &= 2(k - 2j^2 + 2j) \\
\tfrac{1}{2} &= k - 2j^2 + 2j
\end{aligned}
$$

The equation
$k - 2j^2 + 2j$
is an integer by the closture property of integers.

Therefore, we have encountered a contradiction, proving that a counter-example *cannot* exist.

Notes

Implication:
$p \rightarrow q$

Negation:
$p \wedge \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

Getting them mixed up!

3. Getting them mixed up!

A common error students make with direct proofs, proof by contradiction, and proof by contrapositive, is to mix up the steps.

With a proof by contradiction, we come up with an alias for the hypothesis and (negated) conclusion because we're coming up with a hypothetical counter-example.

$$(n)(n) = n^2$$
$$(2j+1)(2j+1) = 2k$$

With a direct proof, we start with the hypothesis, and try to work toward the conclusion.

$$n = 2k+1$$
$$n^2 \Rightarrow (2k+1)^2$$

Notes

Implication:
$p \rightarrow q$

Negation:
$p \wedge \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

3. Getting them mixed up!

A majority of students tend to get this mixed up on exams! Make sure you're paying attention to the steps!

Notes

Implication:
$p \rightarrow q$

Negation:
$p \land \neg q$

Contrapositive:
$\neg q \rightarrow \neg p$

Conclusion

Proof by Contradiction and Proof by Contrapositive can be handy tools when you cannot do a Direct Proof on its own. However, make sure you're not getting all three mixed up!