Mathematical Induction

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About

For these proofs, we will be taking more substantial statements (such as $\sum_{i=1}^{n} (2i-1) = n^2$) and proving them through induction: Showing that the statement holds true for any natural number plugged in.

Topics

1. Closed Formulas, Recursive Formulas, and Sums

2. Closed Formula / Recursive Formula equivalence proofs

3. Closed Formula / Sum equivalence proofs

Closed Formulas, Recursive Formulas, and Sums

Definition: Closed Formula

A closed formula for a sequence is a formula where each term is described only in relation to its <u>position</u> in the list.

Example: List out the first 5 numbers, a_1, a_2, $a_{3,}$ a_4, and a_5.

- $a_n = n + 1$
- $a_n = n^2 + n$

From Discrete Mathematics Mathematical, Ensley & Crawley

Notes

Closed formula:
a formula where each term is described only in relation to its <u>position</u> in the list.

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Example: List out the first 5 numbers, a_1, a_2, $a_{3,}$ a_4, and a_5.

- $a_n = n + 1$ **2, 3, 4, 5, 6**
- $a_n = n^2 + n$ **2, 6, 12, 20, 30**

From Discrete Mathematics Mathematical, Ensley & Crawley

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a formula where each term is described only in relation to its position in the list.

1. Closed, Recursive formulas & Sums

Definition: Recursive Formula (aka recurrence relation)

A recurrence relation is an equation that recursively defines a sequence of values, once one or more initial terms are given. Each further term of the sequence is defined as a function of preceding terms.

Example: List out the first 5 numbers, a_1, a_2, $a_{3,}$ a_4, and a_5.

- $a_1 = 2$ $\qquad a_n = a_{n-1} + 2$

From https://en.wikipedia.org/wiki/Recurrence relation

Notes

Closed formula:
a formula where each term is described only in relation to its position in the list.

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an equation that recursively defines a sequence of values, once one or more initial terms are given. Each further term of the sequence is defined as a function of preceding terms.

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A recurrence relation is an equation that recursively defines a sequence of values, once one or more initial terms are given. Each further term of the sequence is defined as a function of preceding terms.

Example: List out the first 5 numbers, a_1, a_2, a_3, a_4, and a_5.

- $a_1 = 2$ $\quad a_n = a_{n-1} + 2$

$\mathbf{a_2 = 4}$ $\quad \mathbf{a_3 = 6}$ $\quad \mathbf{a_4 = 8}$ $\quad \mathbf{a_5 = 10}$

From https://en.wikipedia.org/wiki/Recurrence relation

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1. Closed, Recursive formulas & Sums

Definition: Summation

For some sequence of numbers a_1, a_2, … a_k, we can use the notation

$$\sum_{k=1}^{n}\left(a_k\right)$$

to denote the sum of the first *n* terms of the sequence.

From https://en.wikipedia.org/wiki/Recurrence relation

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an equation that recursively defines a sequence of values, once one or more initial terms are given. Each further term of the sequence is defined as a function of preceding terms.

Example: Evaluate this sum:

$$\sum_{k=1}^{3}(2k-1)$$

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1. Closed, Recursive formulas & Sums

Example: Evaluate this sum:

$$\sum_{k=1}^{3}(2k-1)$$

k=1	k=2	k=3
2*1 – 1 = 1	2*2 – 1 = 3	2*3 – 1 = 5

So the result is **1 + 3 + 5 = 9.**

From https://en.wikipedia.org/wiki/Recurrence relation

Notes

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1. Closed, Recursive formulas & Sums

Another thing to highlight is that we can break down a sum… These two are equivalent:

$$\sum_{i=1}^{5}(i)$$

$$\sum_{i=1}^{4}(i)+5$$

Notes

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Another thing to highlight is that we can break down a sum… These two are equivalent:

$$\sum_{i=1}^{5}(i) \qquad \sum_{i=1}^{4}(i)+5$$

$a_1 = 1$
$a_2 = 2$
$a_3 = 3$
$a_4 = 4$
$a_5 = 5$

Notes

Closed formula:
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Another thing to highlight is that we can break down a sum… These two are equivalent:

$$\sum_{i=1}^{5}(i)$$

$a_1 = 1$
$a_2 = 2$
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$a_5 = 5$

Sum = 1 + 2 + 3 + 4 + 5 = 15

$$\sum_{i=1}^{4}(i)+5$$

$a_1 = 1$
$a_2 = 2$
$a_3 = 3$
$a_4 = 4$

Sum = 1 + 2 + 3 + 4 + 5 = 15

Notes

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1. Closed, Recursive formulas & Sums

For the second sum, we removed the final term from the summation itself, and made it something we add on separately. We will do this in a proof.

$$\sum_{i=1}^{5}(i)$$

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$a_2 = 2$
$a_3 = 3$
$a_4 = 4$
$a_5 = 5$

Sum = 1 + 2 + 3 + 4 + 5 = 15

$$\sum_{i=1}^{4}(i)+5$$

$a_1 = 1$
$a_2 = 2$
$a_3 = 3$
$a_4 = 4$

Sum = 1 + 2 + 3 + 4 + 5 = 15

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Closed formula:
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Closed Formula /Recursive Formula equivalence proofs

2. Closed/Recursive Formula Equivalence

Show that the sequence defined by the **recursive formula**:

$$a_n = a_{n-1} + 4 \qquad a_1 = 1$$

for $n \geq 2$

is equivalently described by the **closed formula**:

$$a_n = 4n - 3$$

How do we prove this?

Notes

Basis Step:
Check first term.

Inductive Steps:

- Get equation for a_{n-1} using the closed formula
- Plug equation for a_{n-1} into the recursive formula
- Simplify until original statement

2. Closed/Recursive Formula Equivalence

Show that the sequence defined by the **recursive formula**: $a_n = a_{n-1} + 4$ $a_1 = 1$ for $n \geq 2$

is equivalently described by the **closed formula**: $a_n = 4n - 3$

We could plug in numbers to the recursive and closed formula and check…

	n=1	n=2	n=3
Recursive $a_n = a_{n-1} + 4$, $a_1 = 1$	$a_1 = 1$	$a_2 = a_1 + 4$ $= 1 + 4$ $= 5$	$a_3 = a_2 + 4$ $= 5 + 4$ $= 9$
Closed $a_n = 4n - 3$	$a_1 = 4(1) – 3$ $= 1$	$a_2 = 4(2) – 3$ $= 8 – 3$ $= 5$	$a_3 = 4(3) – 3$ $= 12 – 3$ $= 9$

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is equivalently described by the **closed formula**: $a_n = 4n - 3$

But we can't possibly check that for **all possible inputs**! How do we prove that this works for **all numbers (>=2)?**

We use Induction!

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Show that the sequence defined by the **recursive formula**: $a_n = a_{n-1} + 4$ $a_1 = 1$ for $n \geq 2$

is equivalently described by the **closed formula**: $a_n = 4n - 3$

We've looked at terms for n=1, 2, and 3, so next let's look at for n = m-1.

Notes

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Check first term.

Inductive Steps:

- Get equation for a_{n-1} using the closed formula
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is equivalently described by the **closed formula**: $a_n = 4n - 3$

Closed formula:

$a_n = 4n - 3$

$a_{m-1} = 4(m-1) - 3$

$a_{m-1} = 4m - 7$

Now we have an equation for a_{m-1}, so we can plug it into the recursive formula.

Notes

Basis Step:
Check first term.

Inductive Steps:

- Get equation for a_{n-1} using the closed formula
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Now we have an equation for a_{m-1}, so we can plug it into the recursive formula.

Recursive formula:

$a_m = a_{m-1} + 4$

$a_m = 4m – 7 + 4$

Now we have variables we can work with, so we will simplify…

Notes

Basis Step:
Check first term.

Inductive Steps:

- Get equation for a_{n-1} using the closed formula
- Plug equation for a_{n-1} into the recursive formula
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Show that the sequence defined by the **recursive formula**: $a_n = a_{n-1} + 4$ $a_1 = 1$ for $n \geq 2$

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Closed formula:

$\mathbf{a_n = 4n - 3}$

$a_{m-1} = 4(m-1) - 3$

$a_{m-1} = 4m - 7$

Recursive formula:

$a_m = a_{m-1} + 4$

$a_m = 4m - 7 + 4$

$\mathbf{a_m = 4m - 3}$

After simplifying, we have shown that we have converted the Recursive Formula to be of the same form as the Closed Formula. Therefore, this has been proven.

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Show that the sequence defined by the **recursive formula**: $a_n = a_{n-1} + 4$ $a_1 = 1$ for $n \geq 2$

is equivalently described by the **closed formula**: $a_n = 4n - 3$

Note that I changed variables from n to m; I didn't want to write "n = n-1" when I'm plugging n-1 into my equations, so I switched it to "n = m-1" to work with the "m" variable.

Recursive formula:

$a_m = a_{m-1} + 4$

$a_m = 4m - 7 + 4$

$\mathbf{a_m = 4m - 3}$

The variables are interchangible; e.g., if we had f(x) = 2x, then f(w) = 2w. The variable gets swapped out.

Notes

Basis Step:
Check first term.

Inductive Steps:

- Get equation for a_{n-1} using the closed formula
- Plug equation for a_{n-1} into the recursive formula
- Simplify until original statement

Closed Formula / Sum equivalence proofs

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each $n \geq 1$

Again, we have some statement of equivalence. This time, some **sum** is equivalent to some **closed formula**.

n gets plugged in here.

$$\sum_{i=1}^{n}(2i-1)$$

$$a_n = n^2$$

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:

- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each $n\geq 1$

It's still a good idea to prove that it holds for at least n = 1, but maybe also n = 2 and n = 3.

	n=1	n=2	n=3
Sum $\sum_{i=1}^{n}(2i-1)$	= 2*1-1 = 1	= 2*1-1 + 2*2-1 = 1 + 3 = 4	= 2*1-1 + 2*2-1 + 2*3-1 = 1 + 3 = 5 = 9
Closed $a_n=n^2$	$a_1 = 1^2$ = 1	$a_2 = 2^2$ = 4	$a_3 = 3^2$ = 9

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:

- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n\geq 1$

Next will come the inductive part. We will need two separate formulas again: One for

$\sum_{i=1}^{m}(2i-1)$ and another for $\sum_{i=1}^{m-1}(2i-1)$

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:
- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n\geq1$

Sum formula: Equal to the **sum of all values of (2i-1) from i=1 to m-1, plus the final term of the sum at m.**

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

The sum of the first *m-1* items.

The final item in the sum.

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:
- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n\geq 1$

Next, we have to find an equation for $\sum_{i=1}^{m-1}(2i-1)$ to plug into this formula so we can simplify and prove the statement.

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

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Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:

- Write equation for sum
- Find equation for sum to m-1 with closed formula
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3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n\geq 1$

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

Using the original proposition,

$$\sum_{i=1}^{n}(2i-1)=n^2$$

Let's plug in *n = m-1* to get an equation for $\sum_{i=1}^{m-1}(2i-1)$

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:

- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n \geq 1$

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

$$\sum_{i=1}^{n}(2i-1)=n^2$$

$$\sum_{i=1}^{m-1}(2i-1)=(m-1)^2$$

$$\sum_{i=1}^{m-1}(2i-1)=m^2-2m+1$$

Now we have a formula we can plug into the original equation for the sum that we made.

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:

- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n\geq 1$

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

$$\sum_{i=1}^{m-1}(2i-1)=m^2-2m+1$$

Back with this sum, we are going to substitute out the formula for $\sum_{i=1}^{m-1}(2i-1)=m^2-2m+1$

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

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Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:
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Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n \geq 1$

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

$$\sum_{i=1}^{m-1}(2i-1)=m^2-2m+1$$

Back with this sum, we are going to substitute out the formula for $\sum_{i=1}^{m-1}(2i-1)=m^2-2m+1$

$$\sum_{i=1}^{m}(2i-1)=m^2-2m+1+(2m-1)$$

And then we simplify.

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:
- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

3. Closed Formula / Sum Equivalence

Prove that $\sum_{i=1}^{n}(2i-1)=n^2$ for each. $n\geq 1$

$$\sum_{i=1}^{m}(2i-1)=\sum_{i=1}^{m-1}(2i-1)+(2m-1)$$

$$\sum_{i=1}^{m-1}(2i-1)=m^2-2m+1$$

$$\sum_{i=1}^{m}(2i-1)=m^2-2m+1+(2m-1)$$

$$\sum_{i=1}^{m}(2i-1)=m^2-2m+2m+1-1$$

$$\sum_{i=1}^{m}(2i-1)=m^2$$

We have converted it to the form of the original proposition, so therefore we have proven that $\sum_{i=1}^{n}(2i-1)=n^2$

Notes

Base step:
Check for n = 1, n = 2, and n = 3.

Inductive step:

- Write equation for sum
- Find equation for sum to m-1 with closed formula
- Plug m-1 formula into sum formula and simplify.

Conclusion

These proofs are probably the hardest ones in this class. The aren’t a ton of steps, but they can be difficult to memorize, so make sure to get practice!